Bridge principles and combined reasoning

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Abstract

Hume's well-known objection on the purported connections between 'is' and 'ought' in philosophical argumentation gave rise to a quarrel about the legitimacy of statements that bind factualities to norms. The idea of bridge principle has been inaugurated as statements which precisely crystallize principles of mixed reasoning. We critically examine the role of bridge principles within the context of combination of logics, emphasizing situations where such principles spontaneously arise with desirable and undesirable consequences for combined reasoning.

1 Bridge principles and combined logics

As wittily put in [17], nobody knows how many times the passage from David Hume's "A Treatise of Human Nature" (cf. [16], Book 3, Part 1, Section 1, paragraph 27) has been quoted. This passage that generated a never-ending controversy and so many quotations attests that many times people (philosophers, perhaps) draw conclusions involving imperative statements of the form 'ought to be' on the basis of descriptive statements of the form 'what is', and contains the law that from such purely descriptive propositions no prescriptive statements should be deduced.

So, from statements of fact, such as "murder is harmful to society", a statement of obligation such as "you ought not murder" could not be derived. Statements of fact are established through observation and scientific scrutiny. For example, research in sociology could confirm the claim that "murder is harmful to society", but, so it seems that Hume believed, statements of obligation cannot be established through observation or scientific investigation. According to Hume, moral theories are erroneous if they begin with statements of fact and conclude with statements of obligation.

We are not going to add to the count, taking for granted that the passage is sufficiently well-known, but will show, firstly, that Hume's objection precisely identifies the general problem of change of subject matter in logic and argumentation, and secondly, that responding to such objection can serve as a *Leitmotiv* to the (rather) technical area of combinations of logics.

The subject of combinations of logics has the purpose of analyzing (or decomposing) a complex logic into simpler ones, as well as synthesizing logics from its fragments or also composing new logics from previous ones.

As a new and vigorously growing topic in contemporary logic, combinations of logics attend both a pure philosophical interest offered by the possibility of defining mixed logic systems endowed with heterogeneous logic operators, and pragmatical and methodological reasons in Computer Science.

Whether or not Hume is correct and 'ought' cannot be derived from 'is', and whether people reasoning this way commit what has been called the *naturalistic fallacy*, is not only a difficult philosophical question in moral theory but, interestingly enough, it can be seen to permeate several other domains as well, and specifically combining logics.

Most interpreters hold that a derivation linking 'is' to 'ought' is impossible for Hume. Jeremy Bentham, the founder of utilitarianism, was convinced that natural law philosophers commit the mistake against which Hume warns:

"Some fourscore years ago, by David Hume, in his Treatise on Human Nature, the observation was, for the first time, (it is believed,) brought to light – how apt men have been, on questions belonging to any part of the field of Ethics, to shift backwards and forwards, and apparently without their perceiving it, from the question, what has been done, to the question, what ought to be done, and vice versa: more especially from the former of these points to the other. ..."[1]

Although Bentham was aware of Hume's warning and endorsed the view that the right path from a factual statement to an ethical statement should take into account a kind of natural law, it can hardly be doubted that Bentham would not be opposed to using logic to argue about moral statements, as he says explicitly in [4], Chapter 1:

"But enough of metaphor and declamation: it is not by such means that moral science is to be improved".

Other positions, as the one advocated in [17], hold that Hume would be wrong if he thought that people violated this law in ordinary conversation, since his objection did not distinguish between:"Can 'is' imply 'ought'?" from: "Can a change of 'is' imply a change of 'ought'?". By appeal to a certain 'logic of belief changes', [17] suggests that philosophers, as authors of the "systems of morality" referred to in the passage, more often than people engaged in ordinary conversation would be the violators of Hume's law. Therefore logicians, *qua* philosophers, should be careful.

When regarding the question from the domain of combinations of logics, Hume's law concerns the principles that govern combinations of deontic and alethic logics: in order to perform such a jump from 'is' to 'ought', one might appeal to an explicit "bridge principle", which specifically connects 'is' and 'ought'.

The term "bridge principle" appears for the first time in [1] and subsequently (but with some differences in philosophical perspective) in [30], meaning a statement that binds factualities to norms. Thus, for instance,

$$p \to \bigcirc p$$

(where $\bigcirc$ denotes deontic "obligation") is a simple bridge principle representing 'is-ought'which, when explicitly postulated, should suffice to alleviate Hume's objection. The 'ought-implies-can' thesis allegedly attributed

[1]Note at [3], Appendix 4, Section 20.

to Kant[2] is another bridge principle (see below). Although there is a price to pay in assuming bridge principles, they solve such a question in a reasonable and logically acceptable way. Now, an important question is: if bridge principles may be sufficient, are they also *necessary*? In other words, is it true or not that the jump from factualities to norms is *only* possible by explicitly adhering to bridge principles? This is at the same time a basic problem in combining logics and a philosophically motivated question. A propitious research line in this direction is represented by the results in [23].

This question is also relevant for various reasons. The first is that bridge principles may not be necessarily analytical, in the sense that they might not be sentences which are true because of the meaning of their symbols alone. A distinction can be traced, as pointed out by e.g. [23], between logics in the "mathematical sense" and logics in the "philosophical sense", the latter being systems of analytic principles explicating the meanings of logical symbols. So the question involves clarifying which bridge principles are analytic, if such distinction is possible[3].

A second reason for which the above question is pertinent concerns the fact that sometimes bridge principles in a broad sense may appear spontaneously when combining logics. This is a rather intriguing phenomenon, because if indeed bridge principles may not be analytical, the operation of combining logics would be yielding logics outside the philosophical sense. This criticism, however, would be softened by conceding the existence of "almost analytic" bridge principles, as found in [23], Chapter 11.

A formal definition of a bridge principle expressly concerning Hume's maxim is given in [22]: an axiom schema A is a bridge principle if and only if A contains at least one schematic letter which has at least one occurrence within the scope of a deontic "obligation" operator $\bigcirc$, and at least one occurrence outside the scope of any $\bigcirc$. Thus, for instance, the already mentioned 'ought-implies-can' thesis would be formalized through the following bridge principle (according to the above definition):

$$\bigcirc p \to \Diamond p,$$

where the diamond $\Diamond$ denotes the alethic "possibility" operator.

We will, however, widen this definition in the context of combinations of logics in Section 3, and define bridge principles in a general manner (not necessarily for modal logics).

The fact that 'ought', as much as other concepts of deontic logic, is not conveyed as usual intensional operators (such as the alethic ones) but as a modal operator ranging over actions or states of affairs creates some difficulties in the case of the specific bridge principles coming from Hume's criticisms.

Nonetheless a formal treatment is plainly possible within the development of general multimodal logic. Indeed, what is at stake is a bimodal logic, which is properly treated by understanding the mixture of alethic and deontic logics. However, according to some philosophers (who defend Hume's thesis that no non-trivial 'is-ought' deductions are possible)

[2] A controversy related to this topic was raised in [2], who says at page 99: "As a matter of fact, no proposition of that kind [that 'ought implies can'] appears in any of Kant's writings".

[3] In fact, [20] claims that accepting that a distinction could be drawn between analytic and non-analytic is no more than a "metaphysical article of faith".

'is' cannot be logically linked to 'ought' without using combinations of first-order, alethic and deontic logics (cf. e.g. [28] and [23]).

In any case, the point taken here is that combinations of logics and the surprises lurking therein have much to say about Hume's and correlate problems.

It is illustrative to recall Prior's attempts (cf. [19]) on using the apparatus of elementary modal logic to characterize in formal terms the distinction between *normative* and *non-normative* sentences (which enabled him to define senses of 'descriptive content' versus 'normative content'). However, a problem occurs with mixed sentences, which have both descriptive and normative components, and Prior presents us with a paradox: wherever we draw the distinction between non-normative and normative sentences, there appear inferences from non-normative premises to normative conclusions, by merely applying laws of classical propositional logic. So, let us consider the following two inferences involving bridge principles:

(1) "Tea-drinking is common in England. Therefore: Either tea-drinking is common in England or all New Zealanders ought to be shot",

formalized as:

(1') $\mathtt{d} \vdash \mathtt{d} \vee \bigcirc\mathtt{s}$,

and

(2) "Tea-drinking is common in England, or all New Zealanders ought to be shot. Tea-drinking is not common in England. Therefore, all New Zealanders ought to be shot",

formalized as:

(2') $\mathtt{d} \vee \bigcirc\mathtt{s}, \neg\mathtt{d} \vdash \bigcirc\mathtt{s}$.

There are two obvious possibilities: either the mixed sentence $\mathtt{d} \vee \bigcirc\mathtt{s}$ is normative, in which case (1) is an example of an 'is-ought' inference, or it is a non-normative one, in which case (2) would be an example of an 'is-ought' inference. So one of them represents a violation of Hume's thesis in Prior's terms. The conclusion Prior draws from his paradox is that Hume's 'is-ought thesis' is simply false (cf. [19], p. 206): ethical conclusions from premises having no ethical character *can* be derived.

The inferences involved in the paradox may be ethically irrelevant or trivial, as Prior himself recognized, although nobody could find a suitable definition of what it would mean by "ethical irrelevance" or "ethical triviality" attached to an inference.

The paradoxical character of Prior's argument is no unanimity: objections against it have been raised, as for instance in [18].

By using *ad hoc* concepts of combinations of languages and combinations of logics, G. Schurz (cf. [22] and also [23]) was able to state a *generalized Hume's thesis* (*GH*). This kind of treatment is indeed related to the fusion of modal logics (see Section 2).

All this supports our claim that the notion of bridge principle lies at the heart of the activity of combining logics and combining languages. The following sections are devoted to addressing such a relationship.

2 Combining and decomposing logics

The manifold possibilities of combining logics have attracted the attention of logicians, computer scientists and (in a more timid way) philosophers as well, due to different motivations. People from computer science and knowledge representation who are interested in integrating several reasoning modules into a functional, homogeneous one, have developed techniques for combining different logic systems. This allowed, for instance, the combination of temporal, epistemic and alethic logics into a single, but wholly meaningful system. Among the most relevant techniques for combining logics we can mention fusion and product of modal logics, and fibring at a lesser extent.

Despite the pragmatic interest, there are philosophical concerns behind the creation of logics by means of combinations, or, dually, behind the decomposition of a given logic into simpler ones.

It is generally recognized in the debate about pluralism versus monism in logic that, since the beginning of the 20th century, variants of classical logic and non-classical logics have enormously proliferated, with distinct emphasis and different metalogics through diverse methods. In review, there are deontic logics, with their focus on actions; temporal logics, with focus on time issues and also on verbal tenses; modal and multimodal logics, which are almost omnipresent in the formalized philosophical discourse; many-valued logics, destroying the dichotomy between true and false; paraconsistent logics, destroying the tirany of *ex contradictione sequitur quodlibet* (or Principle of Pseudo-Scotus), and so on and so forth.

However, apart from the debate whether there is a unique "correct" logic (monism), or many (pluralism), or none (instrumentalism), there is another concern: how good is the compartimentation of the one logic into several aspects (or the many logics into independent unities)? Does the composition of logics restore the unity, or create even more specimens? It seems clear that philosophers interested in philosophy of logic should take into account what combined logics mean for their arguments.

Since logics can be composed and decomposed, it makes sense to think about the existence of elementary logics, which would be *atomic* in the universe of logics. Thus, these logics could be combined in appropriate ways in order to generate all the familiar logic systems. To the best of our knowledge, this possibility has not yet been deeply explored in the literature.

This section is devoted to briefly summarize some methods for combining logics, which are relevant to the discussion about bridge principles found in Section 3.

Historically, the first methods for combining logics (namely, fusion and products) were developed with the aims of combining modal logics.

The method of *fusion of normal modal logics* was introduced by R. Thomason in [29]. It allows to combine normal modal logics which are presented syntactically (through Hilbert calculi) and semantically (by means of Kripke semantics). The fusion of a modal logic $\mathcal{L}_1$ with a modal logic $\mathcal{L}_2$ is the bimodal logic $\mathcal{L}$, defined over a language with two boxes. The rest of the connectives are assumed to be classical, and so they are shared by $\mathcal{L}_1$ and $\mathcal{L}_2$.

An interesting example of fusion appears in [22], where a first-order alethic-deontic logic is obtained by fusion of a pure alethic logic with a pure deontic logic. This combination is applied to the formal analysis of

Hume's 'is-ought thesis' (see Section 1 above). It is worth noting that the method of fusion was anticipated in the pioneering paper [12] by M. Fitting where, among other intuitively appealing examples, alethic and deontic modalities were fused.

Technically, the fusion at the semantical level of a modal logic $\mathcal{L}_1$ endowed with a box $\Box_1$ with a modal logic $\mathcal{L}_2$ endowed with a box $\Box_2$ is a bimodal logic $\mathcal{L}$ with boxes $\Box_1$ and $\Box_2$ (and the derived diamonds $\Diamond_1$ and $\Diamond_2$) characterized by a general kind of Kripke frames $\langle W, R_1, R_2\rangle$ with a unique set of worlds W and two binary relations R_1 and R_2 over W, one for each modality. The new frames are such that $\langle W, R_1\rangle$ and $\langle W, R_2\rangle$ are Kripke frames for $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. The fusion of the corresponding Hilbert calculi is obtained by simply merging the axioms and inference rules of both logics, keeping in mind that in $\mathcal{L}$ they can be instantiated with mixed formulas (i.e., formulas having one or both boxes).

Another important and early method for combining (modal) logics is the *product of modal logics*. This technique, introduced in [24, 25], is frequently used to represent time-space information because of its nature.

In formal terms, the product of a modal logic $\mathcal{L}_1$ (endowed with a box $\Box_1$) with a modal logic $\mathcal{L}_2$ (endowed with a box $\Box_2$) is, again, a bimodal logic $\mathcal{L}$ with two boxes $\Box_1$ and $\Box_2$; of course, from this it is possible (granted that negation has sufficient properties) to define two diamonds $\Diamond_1$ and $\Diamond_2$, respectively. The logic $\mathcal{L}$ is characterized by all Kripke models of the form

$$\langle W_1 \times W_2, \overline{R}_1, \overline{R}_2, V_1 \times V_2\rangle$$

where, for $i = 1, 2$, $\overline{R}_i \subseteq (W_1 \times W_2) \times (W_1 \times W_2)$ is defined from a relation $R_i \subseteq W_i \times W_i$ as follows:

- $(w_1, w_2)\overline{R}_1(u_1, u_2)$ iff $w_1 R_1 u_1$ and $w_2 = u_2$;
- $(w_1, w_2)\overline{R}_2(u_1, u_2)$ iff $w_2 R_2 u_2$ and $w_1 = u_1$.

On the other hand, $V_1 \times V_2 : \mathbb{P} \longrightarrow \wp(W_1 \times W_2)$ is the mapping given by $(V_1 \times V_2)(p) = V_1(p) \times V_2(p)$, such that $\langle W_i, R_i, V_i\rangle$ is a Kripke model and $V_i : \mathbb{P} \longrightarrow \wp(W_i)$ is a valuation mapping (where $\mathbb{P}$ is the set of propositional variables) for $\mathcal{L}_i$ (for $i = 1, 2$). These processes can be obviously generalized to n-fold products engendering multimodal logics.

A somewhat surprising feature of the product of modal logics is that some new self-generated interactions between modalities (i.e., bridge principles) arise unexpectedly, as we shall discuss in Section 3.

The method of *fibring by functions*, or *fibred semantics*, was first proposed by D. Gabbay in [13] (see also [14]). The original mechanism of fibring was designed to apply for modal logics only. Briefly, the method is defined as follows: given $\mathcal{L}_1$ and $\mathcal{L}_2$ as above, consider Kripke models with distinguished (actual) worlds together with two transfer mappings: h_1 from the set of worlds of the class of models $\mathcal{M}_1$ of $\mathcal{L}_1$ into the class of models $\mathcal{M}_2$ of $\mathcal{L}_2$, and h_2 from the set of worlds of the class of models $\mathcal{M}_2$ of $\mathcal{L}_2$ into the class of models $\mathcal{M}_1$ of $\mathcal{L}_1$. When a Kripke model of $\mathcal{L}_1$ has to evaluate a formula of the form $\Box_2\varphi$ at the actual world w_1, the validity checking is then transferred to the validity checking of $\Box_2\varphi$ within the Kripke model $h_1(w_1)$ at its actual world. The evaluation of a formula of the form $\Box_1\varphi$ within a Kripke model of $\mathcal{L}_2$ at the actual world w_2 is similar, but now using the map h_2. The resulting logic is a normal bimodal logic semantically characterized by the above structures.

Fusion, products, and fibred semantics are valuable tools for combining modal logics. However, being specifically designed for this particular class of logics, they cannot be extended (in an obvious way) to logics of a different nature (see [10] for an adaptation of fibred semantics to matrix logics). So, it is necessary to consider broader techniques if one wants to combine logics in general, but not necessarily modal.

In order to overcome the limitations of fibring by functions, a general definition of fibring using the conceptual tools of category theory was proposed in [26]. This technique is called *categorial fibring* or *algebraic fibring* (cf. [7]).

The idea behind the extension is quite simple: let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two propositional logics to be combined. Suppose, for simplicity, that no connectives are to be shared, that is, the language of the logic $\mathcal{L}$ to be obtained is the free combination of the connectives of both logics. Then, the fibring of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the *least* logic $\mathcal{L}$ defined over the combined language which extends simultaneously $\mathcal{L}_1$ and $\mathcal{L}_2$. In terms of category theory, it is the coproduct of $\mathcal{L}_1$ and $\mathcal{L}_2$ in the underlying category of logics and their morphism (which usually are defined to be logic translations, see below).

Thus, if the logics $\mathcal{L}_1$ and $\mathcal{L}_2$ are presented as Hilbert calculi, the axiomatics of the logic $\mathcal{L}$ obtained by their fibring is the Hilbert calculus defined by merging the rules of both logics, as in the case of fusion. Again, as in the case of fusion, the rules can now be instantiated by mixed formulas of the new language. In particular, if the logics have implications $\to_1$ and $\to_2$, respectively, satisfying *Modus Ponens* then $\mathcal{L}$ will have two implications $\to_1$ and $\to_2$ enjoying *Modus Ponens.*

The above situation may be undesirable (for instance, in the fusion of modal logics) because some connectives of different logics may have similar behavior, and so this technique produces unnecessary duplications of connectives. In order to avoid this, the so-called *constrained fibring* is defined, allowing to share some previously determined connectives. Thus an additional step is to be performed after obtaining $\mathcal{L}$ by the described fibring process: the connectives to be shared are identified in the resulting logic $\mathcal{L}$, therefore reducing the number of replicant connectives and rules. In the example above, just one implication $\to$ (and so just one *Modus Ponens*) will persist in the resulting logic, provided that implication is shared.

Both forms of algebraic fibring (by sharing connectives or not) are universal in the sense of category theory, generalize fusion and fibred semantics at the same time, and can be applied to wider classes of logics such as propositional, modal and higher-order logics.

The morphisms in the categories of logics where fibring is performed are logic translations, that is, mappings between the respective languages which preserve the respective consequence relations. Sometimes, the categorial fibring of two logics computed in categories defined which such a morphisms produces too weak logics; some examples will be given in Section 3. In [9] a category of sequent calculi is proposed in which morphisms preserve sequent rules instead of merely consequence relations. Such rules are a kind of meta-theorems of the logics, and so fibring of logics in this category produces stronger logics than those obtained by fibring using logic translations. For this reason, the categorial fibring within this category is called *metafibring.* An interesting feature of metafibring is that, in general, a logic can be recovered from its fragments.

3 How are spontaneous bridge principles possible?

As we mentioned above with respect to products of logics, an interesting phenomenon occurs in the realm of combination of logics −something which can be regarded as a nuisance, but which also has its positive side: the spontaneous appearance of bridge principles in the resultant of some combined logics.

By "bridge principles" we mean, in a wide sense, any interactions (i.e., derivations) among distinct logic operators which are not instances of valid derivations in the individual logics being combined.

In our sense, bridge principles should not be restricted to correlating ontological to deontological domains (factualities to norms), but should also express connections among different metaphysical perspectives. Why should, for instance, philosophical positions concerning modality be magically reducible to philosophical positions concerning possibility? Accepting any dependence between $\neg\Box\neg p$ and $\Diamond p$ is to accept a bridge principle in the above sense. From the logical point of view, some connections between different connectives (derivations) could be seen as *ad hoc* bridge principles. For instance, $p \vee \neg p$ could be seen as a bridge principle relating the notion of negation with the notion of disjunction which, despite being acceptable from the perspective of classical logic, is not so in a constructive scenery based on intuitionistic logic.

Not all logic interactions are bridge principles: for instance, in the logic $\mathcal{L}$ obtained by combining via metafibring the logic of classical conjunction with the logic of classical disjunction the derivation

$$p \wedge q \vdash (p \wedge q) \vee r$$

is not a bridge principle, as it clearly is obtained by substitution from

$$p \vdash p \vee r$$

which was already valid in the logic of disjunction.

Therefore bridge principles must be the result of reciprocal action or influence of the collective logics being combined, and not merely derived rules or theorems.

A legitimate instance of a bridge principle which emerges spontaneously, returning to the example of combining conjunction with disjunction by metafibring, is

$$p \wedge (q \vee r) \vdash (p \wedge q) \vee r.$$

This is equivalent to the distributivity of $\wedge$ over $\vee$, which, as proven in [5], is not obtained through the logic of conjunction nor through the logic of disjunction (and actually, what is unexpected, cannot be obtained by the usual categorial fibring).

A second case of spontaneous emergence of a bridge principle, now related to (full) classical propositional logic, is the metafibring of the logic of classical negation and the logic of classical disjunction: as shown in [9], the law of excluded-middle $p \vee \neg p$ emerges unavoidably in the combined logic.

In an analogous way, the metafibring of the logic of classical negation and the logic of classical implication brings forth the Principle of Pseudo-Scotus $p \to (\neg p \to q)$ as also shown in [9].

There are two lessons to be drawn from such spontaneous laws that appear in the combination process: the first is that in all three cases the resulting logic (full classical propositional logic in the second and third cases, and the disjunctive-conjunctive fragment of classical propositional logic in the first case) is only recovered from the components thanks to appropriate bridge principles, and, moreover, that such bridge principles are not only expected, but arise inevitably due to the nature of the combination process.

It is worth to emphasize here that, had we used fibring instead of metafibring, no such bridge principles would be obtained in the resulting combined logic.

The second lesson is that self-generated laws may be uncongenial in some cases: it is well-known, for instance, that in the product of two normal modal logics $\mathcal{L}_1$ and $\mathcal{L}_2$ whose languages are endowed with modal operators $\Box_1$ and $\Diamond_1$, and $\Box_2$ and $\Diamond_2$, respectively, the following bridge principles (see, for instance, [15]) are self-generated at the semantic level:

- $(\Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p)$ $\Box$-commutativity;
- $(\Diamond_1\Diamond_2 p \leftrightarrow \Diamond_2\Diamond_1 p)$ $\Diamond$-commutativity;
- $(\Diamond_1\Box_2 p \to \Box_2\Diamond_1 p)$ $(1,2)$-Church-Rosser property;
- $(\Diamond_2\Box_1 p \to \Box_1\Diamond_2 p)$ $(2,1)$-Church-Rosser property.

As a consequence, it follows that other bridge principles will also be derivable, such as:

- $(\Diamond_1^k\Box_2^m p \to \Box_2^m\Diamond_1^k p)$ $(1^k, 2^m)$-Church-Rosser property;
- $(\Diamond_2^k\Box_1^m p \to \Box_1^m\Diamond_2^k p)$ $(2^k, 1^m)$-Church-Rosser property.

Such bridge principles will hardly be regarded as natural; consider, for example, the product of an alethic logic $\mathcal{L}_1$ with an epistemic logic $\mathcal{L}_2$. A form of $(1,2)$-Church-Rosser property as:

$$\Diamond K p \to K \Diamond p$$

will have emerged spontaneously, where K stands for the "knowledge" operator of an specific person. But this is far from acceptable in all cases. Indeed, it is possible that a previously ignorant person can be informed about the following fact $\mathtt{m}$: "there exists an egg-laying mammal" (namely, the platypus). So, it is true that $\Diamond K\mathtt{m}$, but that this does not imply that this person knows *a priori* the possibility of the existence of such an animal which baffled naturalists −in fact, we are assuming that this person did not know even the possibility of $\mathtt{m}$, that is, we are precisely assuming that $K\Diamond\mathtt{m}$ is false. Counterexamples of a similar kind can be given to the other, above mentioned, bridge principles.

It is worth emphasizing that such interactions are only revealed after a careful semantic analysis, and one has no control on which other bridge principles might crop up in certain products. A side effect is that, consequently, it is not possible to obtain *a priori* a complete Hilbert calculus for the product of two modal logics: the bridge principles might have to be explicitly added to the union of the original axiomatics in order to ensure completeness. On the other hand, the convenience of adding appropriate bridge principles adequate to obtain an envisaged logic was already observed by D. Gabbay in [14]. This possibility is sometimes also considered in the context of fusion of logics.

Several other bridge principles can be formulated within modal logics. For example, by taking necessity $\Box$ and possibility $\Diamond$ as primitive operators,

$$\Diamond p \rightarrow \neg\Box\neg p$$

will be a valid principle either in a classical or in an intuitionistic modal logic, whereas its converse

$$\neg\Box\neg p \rightarrow \Diamond p$$

is only classically valid. In the scope of multimodalities a profusion of bridge principles naturally appears (see for instance [8], Chapter 7).

There is another problem concerning bridge principles known as the *collapsing problem*, firstly identified by Gabbay in [13], and later formalized by L. Fariñas del Cerro and A. Herzig in [11]. The latter shows that, by freely combining classical propositional logic and intuitionistic propositional logic the resulting logic collapses to classical logic. The phenomenon arises because both implications collapse, and then intuitionistic implication becomes classic.

Although the collapsing phenomenon can be avoided by some more technically sophisticated processes of combination of logics (see [27] and [6]), the important point is that the collapsing phenomenon is a clear case of undesirable bridge principles:

$$p \rightarrow_c q \vdash p \rightarrow_i q$$

and

$$p \rightarrow_i q \vdash p \rightarrow_c q$$

where $\rightarrow_c$ and $\rightarrow_i$ stand for classical and intuitionistic implication, respectively.

Even if it is possible to escape from the collapsing problem, it is not uninformative: on the contrary, it reveals intrinsic properties of the combination process. But more interesting is to understand *why* certain characteristics of a logic will vanish, transmute or emanate when combined with another. In this regard, we are perhaps closer to the medieval chemical philosophy, than to chemistry when combining logics.

4 The alchemy of transmuting reasoning

More important than the questions of symbolic character is the following question: given logics presented in an homogeneous way, under which conditions is their composition meaningful? Do compositions always make philosophical sense? Such concerns have been already noticed: for example, it was observed in [22] that it is conceivable that some multimodal logics obtained through combination of modal logics by adding arbitrarily chosen bridge principles could turn out to be meaningless.

The examples and cases above present evidence against and in favor of two kinds of situations: in order to avoid the collapsing problem, more careful combination processes should be expected. On the other hand, if one wants to recover a logic from its fragments, a stronger process allowing the generation of bridge principles would be more adequate. Indeed, in this way some intended interactions between the connectives of both logics would guarantee the entirety of the obtained logic. In many cases the choice between careful processes (avoiding excessive interactions between distinct modes of reasoning) and relaxed methods (favoring the emergence of bridge principles) depends upon what one wants to obtain.

Still, there are a number of situations where bridge principles should be deliberately added to the combined logic with compelling consequences in reasoning and argumentation; as recognized in [17] (p. 286), acceptance of bridge principles might even help explain sociological movements.

On another direction, one may think about combining not logics, but abstract structures in the sense of algebra and model theory. This is a completely different account of combined logics (or rather, combined reasoning) where heuristics is privileged and bridge principles would suggest themselves as aims in defining richer structures.

The emergence of spontaneous principles in logic, and besides (as we have seen) depending on the adopted combining principles, seems to defy the restrictive view of Quine (cf. [21]) according to which logic is only first-order, classical predicate logic with identity and his well-known argument against non-standard logics as being no more than a change of language. This is a point that certainly deserves attention from philosophers of logic.

However, it is in the context of modal logics that the issue is more pertinent, and it is, perhaps, no accident that methods for combining logics first manifested themselves in connection with modalities. If Hume is right, it should be a philosophical duty to identify the nuances of change of subject matter in logic and argumentation, and to pay the respective tribute for this. Bridge principles solidify this attitude especially when modalities are concerned.

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Another reason is that as spans increased, engineers were unable to lift larger chains into position, whereas wire strand cables can be formulated one by one in mid-air from a temporary walkway. Suspender cable terminations[edit]. Poured sockets are used to make a high strength, permanent cable termination.Â The principles of suspension used on a large scale also appear in contexts less dramatic than road or rail bridges. Light cable suspension may prove less expensive and seem more elegant for a cycle or footbridge than strong girder supports.Â Inca rope bridge â€" has features in common with a suspension bridge and predates them by at least three hundred years. However, in a rope bridge the deck itself is suspended from the anchored piers and the guardrails are non-structural. We critically examine the role of bridge principles within the context of combination of logics, emphasizing situations where such principles spontaneously arise with desirable and undesirable consequences for combined reasoning. 1. Bridge principles and combined logics. As wittily put in [17], nobody knows how many times the passage from David Humeâ€™s â€œA Treatise of Human Natureâ€ (cf. [16], Book 3, Part 1, Section 1, paragraph 27) has been quoted.