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Research Books:

- R [RB1] “*The Feynman Integral and Feynman's Operational Calculus*”, Oxford Mathematical Monographs, Oxford Science Publications, *Oxford University Press*, Oxford, London and New York, approx. 800 pages (precisely, 771 + (xviii) pages & 21 illustrations), March 2000. ISBN 0 19 853574 0 (Hbk). (With Gerald W. Johnson.) [Corrected Reprinting, Jan. 2001. ***First Paperback Edition***, Jan. 2002. ISBN 0 19 851572 3 (Pbk). Second Reprinting: Jan. 2003. Electronic Edition: forthcoming.] US Library of Congress Classification: QA312.J54 2000.

[This research treatise develops a mathematical theory of the beautiful but challenging subject of the Feynman path integral approach to quantum physics, and of the closely related topic of Feynman's operational calculus for noncommuting operators. It was written over a period of about ten years (Dec. 1989--Dec. 1999) and provides the most complete mathematical treatment of these subjects to date.

Some advantages of the approaches to the Feynman integral which are treated in detail in this book are the following: the existence of the Feynman integral is established for very general potentials in all four cases; under more restrictive but still broad conditions, three of these Feynman integrals agree with one another and with the unitary group from the usual approach to quantum dynamics; these same three Feynman integrals possess pleasant stability properties. The background material in mathematics and physics that motivates the study of the Feynman integral and Feynman's operational calculus is discussed, and detailed proofs are provided for the central results. The last chapter discusses topics in contemporary physics and mathematics (including knot theory and low-dimensional topology) where heuristic Feynman integrals have played a significant role.]

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- R[RB2] “*Fractal Geometry and Number Theory*”. (Subtitle: “*Complex Dimensions of Fractal Strings and Zeros of Zeta Functions*”.) Research Monograph, Birkhäuser-Verlag, Boston, approx. 300 pages (precisely, 268 + (xii) pages & 26 illustrations), January 2000. (With Machiel van Frankenhuysen.) ISBN 0-8176-4098-3. (Boston). ISBN 3-7643-4098-3 (Basel). US Library of Congress Classification: QA614.86.L36 1999.

[This (refereed) research monograph consists entirely of new research carried out by the authors over a period of about five years (March 1995--Nov. 1999). It develops a mathematical theory of ‘complex dimensions’ of fractal strings, and of the oscillations intrinsic to the geometry and the spectrum of the associated fractals. In particular, new results about the critical zeros of zeta functions are established and a geometric reformulation of the (Extended) Riemann Hypothesis is obtained in terms of the notion of complex dimension and of the frequency spectrum of fractal strings. On the fractal side, for example, precise explicit formulas are obtained for the volume of tubular neighborhoods of self-similar (and other fractal) strings.

In the long-term, this work is aimed at putting fractal geometry in an arithmetic context and, conversely, at putting various aspects of number theory (such as the theory of

Dirichlet series, for example) in a geometric framework.]

R[RB3] “*Fractal Geometry, Complex Dimensions and Zeta Functions*”, (Subtitle: “*Geometry and Spectra of Fractal Strings*”.) Refereed Research Monograph, Springer Monographs in Mathematics, Springer-Verlag, New York, approx. 490 pages (precisely, 460 + (xxiv) pages & 54 illustrations), August 2006. (With Machiel van Frankenhuysen.) ISBN-10: 0-387-33285-5. e-ISBN: 0-387-35208-2. ISBN-13: 978-0-387-33285-7. US Library of Congress Control Number: 2006929212.

This is a sequel to and a greatly expanded version of the theory of complex fractal dimensions first developed in [RB2]. It contains a large amount of new research material (including several new chapters, sections, and appendices, along with many new examples and applications), almost all of which is directly connected to the work of the authors (and their collaborators) since the publication of [RB2], and a portion of which appeared in print for the first time in the present book.

Overview (of the approx. 200 pages of new material): New chapters on ‘self-similar flows’ (Chp. 7) and on ‘quasiperiodic patterns of self-similar strings’ (Chp. 3), providing a much more precise understanding (than in [RB2]) of the complex dimensions of (nonlattice) self-similar fractals (in \mathbf{R}) and dynamical systems, as well as of the error terms in the associated ‘explicit formulas’. A new geometric description of self-similar fractal strings (in §2.1.1) and a discussion of self-similar strings with multiple generators (in Chp. 2 and throughout the rest of the book). New (and previously entirely unpublished) section (§6.3.3) on an (operator-valued) Euler product attached to the spectrum of a fractal string (generalizing that for the Riemann zeta function $\zeta(s)$ but also converging in the critical strip $0 < \operatorname{Re} s < 1$); study of the eigenvalue spectrum of the corresponding ‘spectral operator’. New pointwise tube formulas (§8.1.1), with a number of new applications to tube formulas for self-similar strings (in §8.4); new results on average Minkowski content (§8.4.3) and on error terms for nonlattice strings (§8.4.4). New sections (§11.1.1 and §11.4.1) on recent results on zeros of zeta functions in finite arithmetic progressions (extending those of the authors in §9.2 and §9.3 of Chapter 9 of [RB2]). Discussion in the last chapter (Chp. 12) of a number of new topics, including recent results on random fractal strings, quantized strings (fractal membranes), and especially on the complex dimensions of the Koch snowflake curve and its generalizations (via an associated ‘tube formula’), as well as of an outline of a higher-dimensional theory of complex dimensions of self-similar systems and fractals. Also, further discussion of a possible cohomological interpretation of the complex dimensions. New appendix (App. A) on Nevanlinna theory and its applications in this context, and a new section (§B.4 of App. B) on ‘two-variable zeta functions’. New examples, illustrations, theorems, and proofs, scattered throughout the book.]

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Bibliography. Acknowledgements.

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R [RB4] "*In Search of the Riemann Zeros: Strings, fractal membranes and noncommutative spacetimes*", Amer. Math. Soc., Providence, R I, 2008, 600 pages (precisely, 558+(xxix) pp.), February, 2008. ISBN-10: 0-8218-422-5. US Library of Congress Classification: QA333.L37 2007].

[The (physically motivated) theory proposed and developed in this research monograph represents approximately ten years of the author's research on this subject (between about 1996 and 2006). It realizes a synthesis of aspects of string theory, noncommutative geometry, the author (and his collaborators)' theory of fractal strings (now 'quantized' and referred to in this new form as 'fractal membranes') and their complex dimensions, as well as of number theory and arithmetic geometry. In particular, it builds upon and expands—but is also in many ways quite different from—the author's earlier (joint) work in [JA24-27] or in [RB2, RB3]. Much of the material presented in the main part of this book (with

the exception of a portion of Chapter 2 and the first section of Chapter 3) is original and published for the first time.]

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Acknowledgements. Bibliography.

Conventions. Index of Symbols. Author Index. Subject Index.

Refereed Edited Research Books:

- [EB1] **Editor:** "*Progress in Inverse Spectral Geometry*", Proc. Summer School on "Progress in Inverse Spectral Geometry", held in Stockholm (Sweden) in June-July 1994, *Trends in Mathematics Series*, Vol. 1, Birkhäuser-Verlag, Basel, November 1997, 204 pages, (Co-Editor: Stig I. Andersson). ISBN 3-7643-5755-X (Basel). ISBN 0-8176-5755-X (Boston). US Library of Congress Classification: QA614.95.P78 1997.

[This volume gathers original research contributions or survey expository articles by some of the best experts in spectral geometry, along with a few papers by promising junior investigators selected by the editors. With two exceptions, the editors have reviewed and edited each paper individually.]

- [EB2] **Coordinating Editor:** "*Harmonic Analysis and Nonlinear Differential Equations*", (Volume in honor of Prof. Victor L. Shapiro), *Contemporary Mathematics*, Vol. 208, American Mathematical Society, Providence, RI, August 1997, 350 + (xii) pages. (Includes a Dedication and a Preface.) (Co-Editors: Lawrence H. Harper and Adolfo J. Rumbos.) ISBN 0-8218-0565-7 (alk. paper). US Library of Congress Classification: QA403.H223 1997.

[This volume gathers high quality original research contributions or survey expository articles by some of the leading experts in classical and modern analysis, working in the areas of harmonic analysis, nonlinear partial differential and mathematical physics. Each contribution in this volume has been individually refereed according to strict standards set by the American Mathematical Society.]

- [EB3] **Editor:** "*Dynamical, Spectral and Arithmetic Zeta Functions*", Proc. Special Session (Amer. Math. Soc. Annual Meeting, San Antonio, Texas, Jan. 1999), *Contemporary Mathematics*, Vol. 290, American Mathematical Society, Providence, RI, December 2001, 195 + (x) pages. (Co-Editor: Machiel van Frankenhuysen.) ISBN 0-8218-2079-6 (alk. paper). US Library of Congress Classification: QA351.A73 1999.

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[EB4] & [EB5]:

Managing Editor: “*Fractal Geometry and Applications*”. (Subtitle: “*A Jubilee of Benoît Mandelbrot*”). [Two volumes (Parts 1 & 2), totaling approximately 1,100 pages (precisely, 1,080 + (xxvi) pages).] *Proceedings of Symposia in Pure Mathematics* (PSPUM), American Mathematical Society, Vol. 72, Parts 1 & 2, Providence, RI, Dec. 2004. (Co-Editor: Machiel van Frankenhuysen.) ISBN 0-8218-3292-1 (set: acid free paper). ISBN 0-8218-3637-4 (part 1: acid free paper). ISBN 0-8218-3638-2 (part 2: acid free paper). US Library of Congress Classification: QA325.F73 2004.

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"Feynman's Operational Calculus and Beyond".) (With Gerald W. Johnson and Lance Nielsen.)

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