

**Ian M. Musson**

# **Lie Superalgebras and Enveloping Algebras**

**Dedicated to my parents, Eric and Jessie.**

## Preface

The publication of Dixmier's book in 1974 led to increased interest in the structure of enveloping algebras. Considerable progress was made in both the solvable and semisimple cases. For example the primitive ideals were completely classified and much information obtained about the structure of the primitive factor rings [BGR73], [Dix96], [Jan79], [Jan83], [Mat91].

Most of this work was complete by the early 1980's, so it was natural that attention should turn to related algebraic objects. Indeed at about this time some new noncommutative algebras appeared in the work of the Leningrad school led by L. D. Faddeev on quantum integrable systems. The term "quantum group" was used by V. G. Drinfel'd and M. Jimbo to describe particular classes of Hopf algebra that emerged in this way. This subject underwent a rapid development, spurred in part by connections with Lie theory, low dimensional topology, special functions and so on. The algebraic aspects of quantum groups are treated in detail in the books [CP95], [Kas95], [KS97], [Lus93], [Jos95] and [Maj95].

Against this background, Lie superalgebras seem to have been somewhat overlooked. Finite dimensional simple Lie superalgebras over algebraically closed fields of characteristic zero were classified by V.G. Kac in his seminal paper [Kac77a]. However more than thirty years after the classification, the representation theory of these algebras is still not completely understood and the structure of the enveloping algebras of these superalgebras remains rather mysterious.

Nevertheless some fundamental progress has been made. For example the characters of finite dimensional irreducible representations have been determined for Lie superalgebras of types A, Q and recently for the orthosymplectic Lie algebras [Bru03], [Bru04], [CLW09], [GS09], [PS97a], [PS97b], [Ser96]. Moreover we know when the enveloping algebra  $U(\mathfrak{g})$  is a domain, when it has finite global dimension, and some progress has been made on understanding its primitive ideals.

Therefore it is timely, I hope, for a volume that collects together some of what is known about Lie superalgebras and their representations. My original motivation for writing the book was to collect together results that were difficult to find. For this reason I tried to include primarily results that have only appeared in research journals. Of course it is impossible to keep to this rule consistently. No attempt was made to be comprehensive, and I needed to include a certain amount of background material that is well known.

Here is a brief overview of the contents of the book. Chapter 1 contains some basic definitions and the statement of the Classification Theorem for finite dimensional classical simple Lie superalgebras over an algebraically closed field of characteristic zero. Since the proof of the Classification Theorem can be found in the book [Sch79], as well as in the paper of Kac, we do not include the proof here. However we give explicit constructions for each classical simple Lie superalgebra  $\mathfrak{g}$ . This is done in Chapter 2 if  $\mathfrak{g}$  is a close relative of  $\mathfrak{gl}(m, n)$ ,  $\mathfrak{g}$  is orthosymplectic, or  $\mathfrak{g}$  belongs to one of the series P or Q in the Kac classification. The other classical simple Lie superalgebras, which we will call exceptional are dealt with in Chapter 4. In order to construct highest weight modules for  $\mathfrak{g}$ , we need to understand its Borel subalgebras.

Unlike the case of a semisimple Lie algebra, there are in general several conjugacy classes of Borel subalgebras. However, at least if  $\mathfrak{g} \neq \mathfrak{psl}(2, 2)$ , there are only a finite number of conjugacy classes, and we give a combinatorial description of them in Chapter 3. If  $\mathfrak{g} \neq \mathfrak{p}(n), \mathfrak{q}(n)$ , then the choice of a Borel subalgebra  $\mathfrak{b}$  of  $\mathfrak{g}$  leads, in Chapter 5 to a second construction of  $\mathfrak{g}$  as a contragredient Lie superalgebra. This approach is less explicit than the first, and some work is required to reconcile the two points of view. However contragredient Lie superalgebras give a unified approach to several results, in particular to the existence of an even nondegenerate invariant bilinear form on  $\mathfrak{g}$ . An algebra that admits such a form is often called basic.

In Chapter 6 we define the enveloping algebra  $U(\mathfrak{k})$  of a Lie superalgebra  $\mathfrak{k}$ . The study of representations of  $\mathfrak{k}$  is equivalent to that of  $U(\mathfrak{k})$ , and techniques from ring theory can be utilized to investigate  $U(\mathfrak{k})$ . We prove the Poincaré-Birkhoff-Witt (PBW) Theorem using the Diamond Lemma. A crucial difference with the PBW Theorem for Lie algebras is that the basis elements for the odd part of  $\mathfrak{k}$  can only appear with exponents zero or one in a PBW basis.

Let  $\mathfrak{k}$  be a finite dimensional Lie superalgebra over a field and let  $R = U(\mathfrak{k}_0), S = U(\mathfrak{k})$  be the enveloping algebras of  $\mathfrak{k}_0$  and  $\mathfrak{k}$ . By the PBW Theorem  $S$  is finitely generated and free as a left or right  $R$ -module. Suppose that  $S$  is a ring extension of  $R$ , that is  $R$  is a subring of  $S$  with the same 1, and that  $S$  is a finitely generated  $R$ -module. We develop some general methods for studying such ring extensions in Chapter 7. Particular attention is paid to the relationship between prime and primitive ideals in  $R$  and  $S$ . When  $S$  is commutative we have the classical Krull relations of lying over, going up, etc. The usual definitions and proofs do not work well in the noncommutative setting, and we adopt an approach using  $R$ - $S$  bimodules.

In Chapter 8 we set up some of the notation that we will use in subsequent chapters to study the enveloping algebra of a classical simple Lie superalgebra  $\mathfrak{g}$ . Among the topics covered are triangular decompositions  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$  of  $\mathfrak{g}$ , Verma modules and the category  $\mathcal{O}$ . Partitions, which can be used to index a basis for  $U(\mathfrak{n}^\pm)$  are also introduced here. For the rest of this introduction, we assume that  $K$  is an algebraically closed field of characteristic zero, and all Lie superalgebras are defined over  $K$ .

Chapters 9 and 10 are devoted to the study of Verma modules. If  $\mathfrak{k}$  is a semisimple Lie algebra, the homomorphisms between Verma modules were first described by I.N. Bernstein, I.M. Gelfand and S.I. Gelfand, [BGG71] and later in more explicit form by N.N. Šapovalov [Šap72]. We introduce Šapovalov elements for basic simple classical Lie superalgebras. Šapovalov elements for isotropic roots lead to the construction of some new modules which then appear in the Jantzen sum formula.

Classical Schur-Weyl duality provides a deep connection between representations of the symmetric group and representations of the Lie algebra  $\mathfrak{gl}(n)$ . This theory was extended to the Lie superalgebra  $\mathfrak{g} = \mathfrak{gl}(m, n)$  first by Sergeev in [Ser84a], and then in more detail by Berele and Regev [BR87]. In Chapter 11, we give an exposition of this work and also of a beautiful extension of the Robinson-Schensted-Knuth correspondence from [BR87]. This correspondence allows us to use semistandard tableaux to index a basis for the simple  $U(\mathfrak{g})$ -modules that appear in the decomposition of the tensor powers of the defining representation of  $\mathfrak{g}$ .

In the theory of symmetric polynomials a key role is played by Schur polynomials. These can be defined in three different ways: as a quotient of alternants, using the Jacobi-Trudi identity, and by using semistandard tableaux. The first definition is related to the Weyl character formula. Our treatment of supersymmetric polynomials in Chapter 12 places particular emphasis on super Schur polynomials. Each definition of the usual Schur polynomials can be extended to the super case, and the main results, due to Pragacz and Thorup [PT92], [Pra91] and Remmel [Rem84], demonstrate the equivalence of the extended definitions. There is a connection with Schur-Weyl duality since the characters of composition factors of tensor powers of the defining representation of  $\mathfrak{gl}(m, n)$  are given by super Schur polynomials.

Chapter 13 is devoted to the center  $Z(\mathfrak{g})$  of  $U(\mathfrak{g})$  and related topics. Denote the fixed ring of  $S(\mathfrak{h})$  under the action of the Weyl group  $W$  by  $S(\mathfrak{h})^W$ . There is an injective algebra map from  $Z(\mathfrak{g})$  to  $S(\mathfrak{h})^W$  which we call the Harish-Chandra homomorphism. Unlike the case of semisimple Lie algebras however, this map is not surjective, but its image can be explicitly described. This result was first formulated by Kac [Kac84], but a gap in the proof was later filled by Gorelik [Gor04] and independently by the author (unpublished). On the other hand, Sergeev [Ser99a] proved a version of the Chevalley restriction theorem for basic classical simple Lie superalgebras, [Ser99a]. This can be used to give another proof of the theorem formulated by Kac, but we will in fact deduce Sergeev's Theorem from the result about the center. If  $\mathfrak{g} = \mathfrak{gl}(m, n)$  or an orthosymplectic Lie superalgebra, supersymmetric polynomials can be used to give an explicit set of generators for the image of the Harish-Chandra homomorphism, and to describe the central characters of  $\mathfrak{g}$ .

In Chapter 14 we study finite dimensional modules for a basic classical simple Lie superalgebra. If  $\mathfrak{g}$  is a close relative of  $\mathfrak{gl}(m, n)$  or  $\mathfrak{g}$  is orthosymplectic, we give necessary and sufficient conditions for a simple highest weight module to be finite dimensional in terms of the highest weight. Then we prove the Kac-Weyl character formula for finite dimensional typical modules.

If  $\mathfrak{k}$  is a finite dimensional Lie algebra, then the space of primitive ideals  $\text{Prim } U(\mathfrak{k})$  is now well understood in both the solvable and semisimple cases. By comparison much less is known about primitive ideals in the enveloping algebra of a Lie superalgebra  $\mathfrak{g}$ , but in Chapter 15 we survey what is known, with particular emphasis on the case where  $\mathfrak{g}$  is classical simple. To do this it is convenient to review the semisimple Lie algebra case.

Unlike the Lie algebra case, the enveloping algebra  $U(\mathfrak{k})$  of a Lie superalgebra  $\mathfrak{k}$  may contain zero divisors. This will be the case if  $\mathfrak{k}$  contains a non-zero odd element  $x$  such that  $[x, x] = 0$ . In the absence of such elements  $\mathfrak{k}$  is called *torsion-free*. In Chapter 17 we prove a theorem of R. Bøgvad, stating that the enveloping algebra of a torsion-free Lie superalgebra is a domain, [Bøg84], see also [AL85]. The proof of this result, in contrast to the simplicity of its statement, requires a considerable amount of homological algebra.

In Chapter 16 we develop the necessary cohomology theory of Lie superalgebras. This includes the fact that if  $M$  is a  $\mathbb{Z}_2$ -graded  $\mathfrak{k}$ -module, then the even part of  $H^2(\mathfrak{k}, M)$  parameterizes extensions of  $M$  by  $\mathfrak{k}$ , the cup product in cohomology, and the Hochschild-Serre spectral sequence. We give a self-contained account of the

necessary background on spectral sequences. The cohomology of a Lie superalgebra  $\mathfrak{k}$  can be computed using the standard resolution of the trivial  $\mathfrak{k}$ -module. In contrast to the Lie algebra case, this resolution has infinitely many terms if  $\mathfrak{k}_1 \neq 0$ . Chapter 17 deals with the more ring theoretic aspects of homological algebra needed to prove Bøgvad's result. These include standard results on derived functors and global dimension, as well as the Yoneda product, the bar resolution and the Löffwall algebra. In the final chapter we introduce affine Lie superalgebras and obtain some applications to number theory. These applications concern the number of ways to write an integer as sum of a given number of squares, or as sum of a given number of triangular numbers. The main results here are due to Kac and Wakimoto [KW94] and Gorelik [Gor09a], [Gor09b]. Some background material on Lie theory, Hopf algebras and ring theory is given in Appendix A.

This book has grown to about twice the length I originally intended, and unfortunately some important results had to be left out. However some of the topics not covered here are treated in other texts. Connections with Physics are dealt with in [DM99] and [Var04]. The Dictionary of Lie Algebras and Lie Superalgebras [FSS00], while not containing any proofs, is nevertheless an invaluable source for detailed information about Lie superalgebras and their representations. We also recommend the survey article of Serganova on affine Lie superalgebras and integrable representations [Ser09]. On the other hand many topics are included which I believe can be found in no other texts. These topics include, in the order they appear in the book, the construction of the exceptional Lie superalgebras, many of the ring theoretic methods used to study enveloping algebras, material on Schur-Weyl duality, supersymmetric polynomials, the center and central characters, the question of when the enveloping algebra contains zero divisors, and applications of affine Lie superalgebras to number theory. The treatment of Borel subalgebras that we give here is probably new.

I have used parts of this book to teach courses at the University of Wisconsin-Milwaukee and elsewhere. Of course it works best for students with some background in Lie theory. Here are some suggestions about how to use this book as a textbook. Chapter 1 contains the basic definitions, so is a prerequisite for everything else. Then Chapters 2-5 form a basic course on Lie superalgebras. Chapters 6 - 8, possibly followed by parts of Chapters 9, 10, 13, 14, 15 could be used for a course dealing with enveloping algebras. For more combinatorics use Chapters 11 and 12, for homological topics Chapters 16 and 17, and for applications to number theory Chapter 18. Exercises are given at the end of each Chapter, often providing examples to illustrate the theory.

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There exist two types of basic classical Lie superalgebras (excluding Lie algebras), see Chapters 2 and 4 in [31] for the explicit definition of the Lie superalgebras we introduce. For type I, the adjoint representation of  $\mathfrak{g}_0$  in  $\mathfrak{g}_1$  decomposes into two irreducible representations. The list of basic classical Lie superalgebras of type II consists of  $\text{osp}(m|2n)$  for  $m = 2, D(2, 1; \hat{1} \pm), F(4)$  and  $G(3)$ . An arbitrary Borel subalgebra of  $\mathfrak{g}$  (see Chapter 3 of [31]) will be denoted by  $\mathfrak{b}$ . Since these Borel subalgebras are not always conjugate under the action of the Weyl group Kac-Moody algebras and superalgebras, two-dimensional conformal symmetry and its extensions. When completed, we have in mind to gather the three parts in a unique volume. 25 Matrix realizations of the classical Lie superalgebras 26 Nilpotent and solvable Lie superalgebras 27 Orthosymplectic superalgebras 28 Oscillator realizations: Cartan type superalgebras 29 Oscillator realizations: orthosymplectic and unitary series 30 Oscillator realizations: strange series 31 Real forms 32 Representations: basic definitions 33 Representations: exceptional superalgebras 34 Representations: highest weight representations 35 Representations: induced modules 36 Representations: orthosymplectic superalgebras 37 Representations: reducibility 38. Recently, Yau laid the foundation of a homology theory for Hom-Lie algebras and constructed the enveloping algebras of Hom-Lie and Hom-Leibniz algebras in [17]-[19]. Many more properties and structures of Hom-Lie algebras have been developed (see [20]-[23] and references cited therein). In [33], Ammar and Makhoulouf introduced the notion of Hom-Lie superalgebras and provided a construction theorem from which one can derive a one parameter family of Hom-Lie superalgebras deforming the orthosymplectic Lie superalgebras.