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Some Balanced Incomplete Block Designs

A. L. WHITEMAN*
Department of Mathematics
University of Southern California
Los Angeles, CA 90089-1113, U.S.A.

Dedicated to Solomon Golomb on the occasion of his 60^{th} birthday

Abstract—New constructions for SBIBDs with parameters $(v, k, \lambda) = (45, 12, 3)$, $(36, 15, 6)$, and $(40, 13, 4)$ are given in this note.

1. INTRODUCTION

A symmetric balanced incomplete block design SBIBD with parameters v, k, λ can be defined as a square $(0,1)$-matrix of order v with k 1s in each row and column and with the inner product of a pair of distinct rows equal to λ. In Table I of the first edition of his book *Combinatorial Theory*, Hall [1] lists as unknown designs with parameters $(v, k, \lambda) = (36, 15, 6)$ and $(45, 12, 3)$. Today many constructions for these designs are known. A survey by Kingsley and Stanton [2] summarizes known designs and presents partial isomorphism results. The constructions in the present note have the interesting feature of being closely interconnected.

2. PRELIMINARY IDENTITIES

The constructions employ four symmetric $(0,1)$-matrices N_0, N_1, N_2, N_3 of order 9. The matrices are as follows:

$$N_0 = \left|\begin{array}{c|c|c} 010 & 000 & 001 \\ 100 & 000 & 001 \\ 000 & 110 & 000 \\ \hline 001 & 010 & 000 \\ 001 & 100 & 000 \\ 000 & 000 & 110 \\ \hline 000 & 001 & 010 \\ 000 & 001 & 100 \\ 110 & 000 & 000 \end{array}\right|, \qquad N_1 = \left|\begin{array}{c|c|c} 001 & 000 & 010 \\ 000 & 101 & 000 \\ 100 & 000 & 010 \\ \hline 010 & 001 & 000 \\ 000 & 000 & 101 \\ 010 & 100 & 000 \\ \hline 000 & 010 & 001 \\ 101 & 000 & 000 \\ 000 & 010 & 100 \end{array}\right|,$$

*Deceased.

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$$N_2 = \left|\begin{array}{c|c|c} 000 & 100 & 100 \\ 000 & 010 & 010 \\ 000 & 001 & 001 \\ \hline 100 & 000 & 100 \\ 010 & 000 & 010 \\ 001 & 000 & 001 \\ \hline 100 & 100 & 000 \\ 010 & 010 & 000 \\ 001 & 001 & 000 \end{array}\right|, \qquad N_3 = \left|\begin{array}{c|c|c} 000 & 011 & 000 \\ 001 & 000 & 100 \\ 010 & 000 & 100 \\ \hline 000 & 000 & 011 \\ 100 & 001 & 000 \\ 100 & 010 & 000 \\ \hline 011 & 000 & 000 \\ 000 & 100 & 001 \\ 000 & 100 & 010 \end{array}\right|.$$

These matrices may be regarded as circulant matrices of order 3 in which the entries themselves are square matrices of order 3. We shall require a number of identities.

It may be verified by direct inspection that

$$I + N_0 + N_1 + N_2 + N_3 = J, \tag{1}$$

where I is the identity matrix of order 9 and J is a square matrix of order 9 with all entries equal to 1. The four matrices N_i are also interconnected by the interesting relation

$$N_i N_j = N_k + N_l, \tag{2}$$

where the subscripts i, j, k, l constitute any permutation of the integers $0, 1, 2, 3$. For example,

$$N_0 N_1 = N_2 + N_3.$$

It follows from (1) and (2) that

$$(I + N_j)(I + N_k) = J, \qquad (j \neq k). \tag{3}$$

By squaring both sides of (1) and simplifying with the aid of (2), we deduce the identities

$$(I + N_0)^2 + (I + N_1)^2 + (I + N_2)^2 + (I + N_3)^2 = 9I + 3J \tag{4}$$

and

$$(J - (I + N_0))^2 + (I + N_1)^2 + (I + N_2)^2 + (I + N_3)^2 = 9I + 6J. \tag{5}$$

3. CONSTRUCTION OF A SBIBD WITH PARAMETERS (45,12,3)

Let O denote a square matrix of order 9 in which each entry is zero. Put $M_i = I + N_i$ $(i = 0, 1, 2, 3)$. We shall prove that the matrix

$$A = \left|\begin{array}{c|c|c|c|c} O & M_0 & M_1 & M_2 & M_3 \\ \hline M_3 & O & M_0 & M_1 & M_2 \\ \hline M_2 & M_3 & O & M_0 & M_1 \\ \hline M_1 & M_2 & M_3 & O & M_0 \\ \hline M_0 & M_1 & M_2 & M_3 & O \end{array}\right|$$

is a SBIBD with parameters $v = 45$, $k = 12$, $\lambda = 3$.

The number of 1s in each of the nine rows (columns) of M_i $(i = 0, 1, 2, 3)$ is three. Consequently, the number of 1s in each of the 45 rows (columns) of A is 12.

Let $AA^{\top}$ be partitioned into square blocks of order 9 in the same way as A. The identity in (4) implies that the entry in each diagonal block of $AA^{\top}$ is $9I + 3J$. In view of (3), the entry in each off-diagonal block of $AA^{\top}$ is $3J$. It follows that the inner product of any two of the 45 rows of A is equal to 3.

The proof that A is a SBIBD with parameters (45,12,3) is thus complete.

4. CONSTRUCTION OF A SBIBD WITH PARAMETERS (36,15,6)

We shall prove that the matrix

$$B = \begin{array}{|c|c|c|c|} J-M_0 & M_1 & M_2 & M_3 \\ \hline M_3 & J-M_0 & M_1 & M_2 \\ \hline M_2 & M_3 & J-M_0 & M_1 \\ \hline M_1 & M_2 & M_3 & J-M_0 \end{array}$$

is a SBIBD with parameters $v = 36$, $k = 15$, $\lambda = 6$.

The number of 1s in each of the 36 rows (columns) of B is 15.

Let $BB^\top$ be partitioned into square blocks of order 9 in the same way as B. The identity in (5) implies that the entry in each diagonal block of $BB^\top$ is $9I + 6J$. In view of (3), the entry in each off-diagonal block of $BB^\top$ reduces to $6J$. Therefore, the inner product of any two of the 36 rows of B is equal to 6.

This completes the proof that B is a SBIBD with parameters $v = 36$, $k = 15$, $\lambda = 6$.

5. CONSTRUCTION OF A SBIBD WITH PARAMETERS (40,13,4)

The identities in Section 2 provide us with a nice bonus. In the following configuration:

$$\begin{array}{|c|c|c|c|c|} M_0 & M_1 & M_2 & M_3 & B_0 \\ \hline M_3 & M_0 & M_1 & M_2 & B_1 \\ \hline M_2 & M_3 & M_0 & M_1 & B_2 \\ \hline M_1 & M_2 & M_3 & M_0 & B_3 \\ \hline B_0^\top & B_1^\top & B_2^\top & B_3^\top & J \end{array},$$

each B_i ($i = 0, 1, 2, 3$) has nine rows and four columns. The elements in the i^{th} column of B_i are all 1s and the remaining elements are all 0s. J is a square matrix of order 4 with all entries equal to 1.

We conclude with an exercise for the reader. Verify that the configuration above is a SBIBD with parameters $v = 40$, $k = 13$, $\lambda = 4$.

REFERENCES

1. M. Hall, Jr., *Combinatorial Theory*, Blaisdell, Waltham, MA, (1967).
2. R.A. Kingsley and R.G. Stanton, A survey of certain balanced incomplete block designs, *Proceedings of the Third Southeastern Conference on Combinatorics, Graph Theory and Computing*, Boca Raton, FL, pp. 305–310, (1972).

This paper describes and lists certain block designs for two non-interacting sets of treatments. The designs have the following properties: (a) each set of treatments is arranged relative to blocks in a non-symmetrical balanced incomplete block design, the same balanced incomplete block solution being used for both sets of treatments; (b) each set of treatments is totally balanced with respect to the other, i.e. each is disposed relative to the other in the same way. This module gathers everything related to Balanced Incomplete Block Designs. One can build a BIBD (or check that it can be built) with balanced_incomplete_block_design(): sage: BIBD = designs.balanced_incomplete_block_design(7,3). In particular, Sage can build a \((v,k,1)\)-BIBD when one exists for all \(k\leq 5\). The following functions are available: balanced_incomplete_block_design(). Return a BIBD of parameters \(v,k\). BIBD_from_TD().Â R. H. F. Denniston, Some maximal arcs in finite projective planes. Journal of Combinatorial Theory 6, no. 3 (1969): 317-319. doi:10.1016/S0021-9800(69)80095-5. sage.combinat.designs.bibd.BIBD_from_difference_family(G, D, lambd=None, check=True)Â¶.