TWO DEPENDENCE MEASURES FOR MULTIVARIATE EXTREME VALUE DISTRIBUTIONS

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Outline:

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- Dependence measures τ_1, τ_2

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- Relations between τ_1, τ_2

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1. Introduction

$$\mathbf{X} = (X_1, X_2, \cdots, X_d) \sim G(\mathbf{x}), \quad \mathbf{x} \in \mathbf{R}^d,$$

where G is a multivariate extreme value distribution function. WOLOG with Fréchet margins: for all j

$$G_j(x) = \exp\{-1/x\} \quad (x > 0)$$

and exponent function

$$\lambda(\mathbf{x}) = -\log G(\mathbf{x}),$$

$$G^t(t\mathbf{x}) = G(\mathbf{x}) \quad \Rightarrow \quad t\lambda(t\mathbf{x}) = \lambda(\mathbf{x}), \ (t > 0).$$

Since for MEVD

$$\min_{1 \le j \le d} G_j(x_j) \ge G(\mathbf{x}) \ge \prod_{j=1}^{d} G_j(x_j),$$

in our case

$$\max \frac{1}{x_j} \le \lambda(\mathbf{x}) \le \sum \frac{1}{x_j}.$$

(complete dependence) (total independence)

The homogeneity of λ implies that $\lambda(tx)/\Sigma(tx_j)^{-1}$ does not depend on t. Define the (generalized) Pickands dependence function

$$A(\mathbf{v}) = \lambda(v_1^{-1}, v_2^{-1}, \cdots, v_d^{-1}) \quad \mathbf{v} \in \Omega,$$

$$\text{where} \qquad \Omega = \{\mathbf{v} : v_j \geq 0,\ \Sigma v_j = 1\}$$

is the unit-simplex. It follows that

$$\frac{1}{d} \leq A_0(\mathbf{v}) =: \max v_j \leq A(\mathbf{v}) \leq 1,$$

$$\lambda(\mathbf{x}) = A(\mathbf{v})\Sigma x_j^{-1},$$

where $v_j = x_j^{-1}/\Sigma x_i^{-1}$.

$$\eta = A\left(\frac{1}{d}, \cdots, \frac{1}{d}\right)$$

has an interesting interpretation:

$$P\left\{\max_{1\leq j\leq d} X_j \leq z\right\} = \exp\{-\lambda(z, z, \cdots, z)\}$$

$$= \exp\{-d\eta/z\} = \{\exp\{-1/z\}\}^{d\eta}.$$

Hence, $\theta = d\eta$ is the *extremal coefficient* of $(X_1, X_2, \cdots, X_d)$.

$$\theta = 1 \Leftrightarrow \text{complete dependence}$$

$$\theta = d \Leftrightarrow \text{total independence}$$

Schlather and Tawn (2002) analyse $\theta_B = |B|\eta_B$ for all 2^d possible subsets B of $\{1, 2, \cdots, d\}$.
From de Haan and Resnick (1977) and Pickands (1981)

$$A(\mathbf{v}) = \int_\Omega \max v_j a_j U(d\mathbf{a})$$

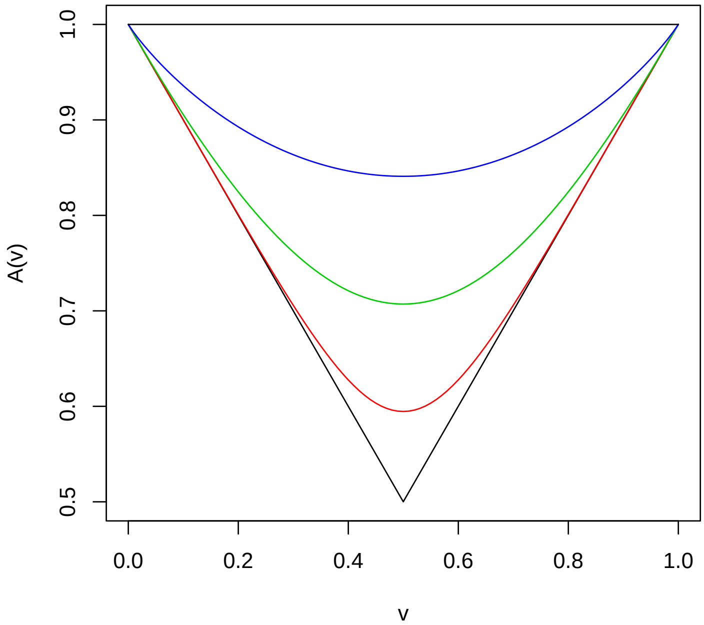
for a finite positive measure U,

$$U(\Omega) = d \text{ and } \int_\Omega a_j U(d\mathbf{a}) = 1 \text{ for all } j.$$

The function A is convex because for $0 \le \alpha \le 1$,

$$\max\{(\alpha v_j + (1-\alpha) w_j) a_j\}$$

$$\le \alpha \max v_j a_j + (1-\alpha) \max w_j a_j.$$



Pickands dependence function for the Logistic Model $A(v) = (v^{1/\alpha} + (1-v)^{1/\alpha})^\alpha$ with $\alpha = 0,\ .25,\ .50,\ .75,\ 1.$

2. Measures of Dependence: Rescaling η, a natural measure of dependence is

$$\tau_1 = \frac{1 - A\left(\frac{1}{d}, \cdots, \frac{1}{d}\right)}{\max_A \left\{1 - A\left(\frac{1}{d}, \cdots, \frac{1}{d}\right)\right\}}$$

$$= \frac{d-\theta}{d-1} = \frac{d}{d-1}(1-\eta)$$

An alternative measure is

$$\tau_2 = \frac{\int_\Omega (1 - A(\mathbf{v}))d\mathbf{v}}{\max_A \int_\Omega (1 - A(\mathbf{v}))d\mathbf{v}}$$

$$= \frac{\int_\Omega (1 - A(\mathbf{v}))d\mathbf{v}}{\int_\Omega (1 - A_0(\mathbf{v}))d\mathbf{v}} =: \frac{S_d(A)}{S_d(A_0)}.$$

Which one is preferred?

Similar question: mode vs. mean.

Expect from dependence measure that for

$$A = \alpha A_0 + (1-\alpha) \cdot 1$$

$$\Rightarrow \qquad \tau = \alpha.$$

Indeed, for this **mixture model**

$$\tau_1 = \tau_2 = \alpha.$$

To compute τ_2 we need a formula for S_{A_0}, the volume above A_0:

d	S_{A_0}
2	$1/4 = .2500$
3	$7/36 = .19444$
4	.07986
5	.02264

Until very recently the challenge was to find a formula for $S_d(A_0)$. My colleague **Shmuel Onn** derived and proved

$$\boxed{S_d(A_0) = \frac{1}{(d-1)!} - \frac{B_d}{d!}}$$

where

$$B_d = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{d}\right)$$

is the harmonic sum.

Other (bivariate) measures of dependence:

In the literature (Beirlant et al, 2004) we encounter

$$\tau_K = \text{Kendall's tau} = 4EC(G_1(X_1), G_2(X_2)) - 1,$$

$$\rho_S = \text{Spearman's rho} = corr(G_1(X_1), G_2(X_2)),$$

$$\rho = corr(\log G_1(X_1), \log G_2(X_2)).$$

Tawn (1988) mentioned τ_1 for $d = 2$. I have not seen τ_2. These are all marginal-free and for mixture distributions (not mixture exponents):

$$(X_1, X_2) = \left\{ \begin{array}{lll} (U, V) & \text{w.p.} & 1 - \alpha \\ (U, U) & \text{w.p.} & \alpha \end{array} \right\}$$

U, V independent,

$$\tau_K = \rho_S = \rho = \alpha.$$

3. Examples.

Let $V_1, V_2, \cdots$ be i.i.d. unit-Fréchet.

Mixture model: For $0 \le \alpha \le 1$

$$\lambda(x, y) = \alpha \max(x^{-1}, y^{-1}) + (1 - \alpha)(x^{-1} + y^{-1}).$$

That is

$$X = \max(\alpha V_1, (1 - \alpha) V_2)$$

$$Y = \max(\alpha V_1, (1 - \alpha) V_3).$$

$$A(v) = \alpha \max(v, 1 - v) + (1 - \alpha) \cdot 1 \quad (v \in [0, 1]).$$

$$\tau_1 = \tau_2 = \alpha.$$

$$\tau_K = \rho = \frac{\alpha}{2 - \alpha} \le \rho_S = \frac{3\alpha}{4 - \alpha} \le \alpha.$$

$\alpha = \tau_1 = \tau_2$	ρ_S	$\tau_K = \rho$
0	0	0
1/4	1/5	1/7
1/2	3/7	1/3
3/4	9/13	3/5
1	1	1

Mixed model:

$$\lambda(x,y) = \frac{1}{x} + \frac{1}{y} - \frac{\alpha}{x+y}$$

$$A(v) = 1 - \alpha(1-v)v$$

$$\tau_1 = \frac{\alpha}{2}, \quad \tau_2 = \frac{2}{3}\alpha$$

$$\tau_K = \frac{8tan^{-1}(\alpha/(4-\alpha))^{1/2}}{\alpha^{1/2}(4-\alpha)^{1/2}} - 2$$

$$\rho = \frac{8tan^{-1}(\alpha/(4-\alpha))^{1/2}}{\alpha^{1/2}(4-\alpha)^{3/2}} - \frac{2-\alpha}{4-\alpha}$$

$$\rho_S = 12\left\{\frac{8tan^{-1}(\alpha/(8-\alpha))^{1/2}}{\alpha^{1/2}(8-\alpha)^{3/2}} + \frac{1}{8-\alpha}\right\} - 3$$

α	τ_K	ρ	τ_1	ρ_S	τ_2
0	0	0	0	0	0
.25	.0877	.0901	.1250	.1299	.1667
.50	.1853	.1958	.2500	.2702	.3333
.75	.2947	.3215	.3750	.4222	.5000
1	.4184	.4728	.5000	.5874	.6667

de Haan - Resnick model:

$$\lambda(x,y,z) = \frac{1}{2}\{\max(x^{-1},y^{-1}) + \max(x^{-1},z^{-1}) + \max(y^{-1},z^{-1})\}$$

$$X_1 = \max(V_1,V_2)/2$$

$$X_2 = \max(V_1,V_3)/2$$

$$X_3 = \max(V_2,V_3)/2$$

$$A(\mathbf{v}) = \frac{1}{2}\{\max(v_1,v_2) + \max(v_1,v_3) + \max(v_2,v_3)\}$$

$$\eta = A(1/3,1/3,1/3) = 1/2, \;\; \tau_1 = (3/2)(1-\eta) = 3/4$$

$$\tau_2 = \frac{36}{7}\cdot\frac{1}{8} = \frac{9}{14} = .642857$$

$$\tau_1(1,2) = \tau_2(1,2) = 1/2$$

(Introducing X_3 to the system increases the dependence)

Non-symmetric model:

$$X_1 = \max(V_1/2, V_2/4, V_3/4)$$

$$X_2 = \max(2V_1/3, V_2/3)$$

$$X_3 = V_3$$

$$A(\mathbf{v}) = \max(.75v_1, v_2) + \max(.25v_1, v_3)$$

$$(v_1 + v_2 + v_3 = 1)$$

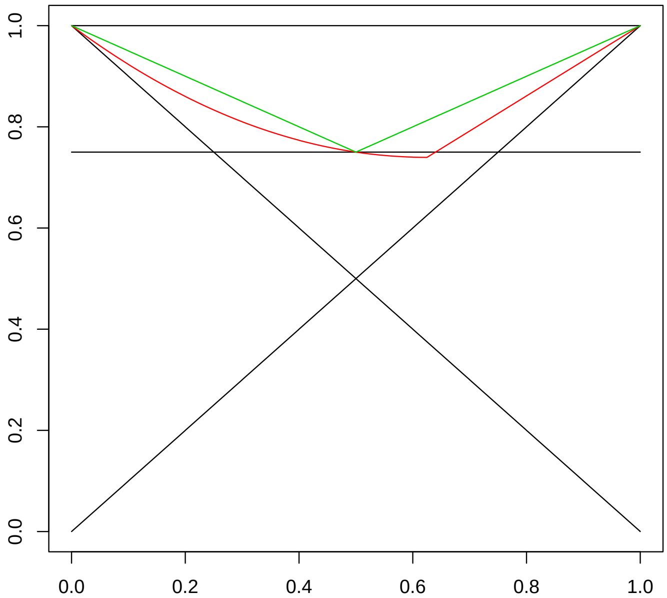
$$\eta = 2/3, \;\; \tau_1 = 1/2, \;\; \tau_2 = (36/7).104762 = .53876$$

$$\tau_1(1,2) = 3/4 = .75 \quad \tau_2(1,2) = 6/7 = .85714$$

$$\tau_1(1,3) = 1/4 = .25 \quad \tau_2(1,3) = 4/10 = .4$$

4. Relations between τ_1 and τ_2.

Theorem. For $d = 2$, $\tau_1 \le \tau_2$.



Proof. Denote $1 - h = A(1/2, 1/2))$,

$$\Rightarrow \qquad \tau_1 = 2h.$$

Define the mixture model (green graph)

$$A^*(\mathbf{v}) = \tau_1 A_0(\mathbf{v}) + 1 - \tau_1,$$

$$\Rightarrow \qquad \tau_1^* = \tau_1 = \tau_2^*.$$

Since A is convex, $A \le A^*$(= at $(1/2, 1/2)$),

$$\int_\Omega (1 - A) \ge \int_\Omega (1 - A^*) = \tau_1 \int_\Omega (1 - A_0)$$

$$\tau_2 = \frac{\int_\Omega (1 - A)}{\int_\Omega (1 - A_0)} \ge \tau_1.$$

This is a perfect proof for $d = 2$. For $d \ge 3$, the picture is misleading, namely, $A \le A^*$ is not necessarily true. Here is a counter example: de Haan-Resnick model.

For $v_1 \ge v_2 \ge v_3, \;\; v_1 + v_2 + v_3 = 1$,

$$A(\mathbf{v}) = v_1 + \frac{v_2}{2}, \;\; A^*(\mathbf{v}) = \frac{3}{4} v_1 + \frac{1}{4}.$$

Since $v_2 \ge v_3 \Leftrightarrow v_2 \ge (1 - v_1)/2$,

$$A(\mathbf{v}) - A^*(\mathbf{v}) = \frac{1}{4}v_1 + \frac{1}{2}v_2 - \frac{1}{4} \geq 0,$$

with equality when $v_1 \geq 1/3,\ v_2 = v_3 = (1 - v_1)/2$.

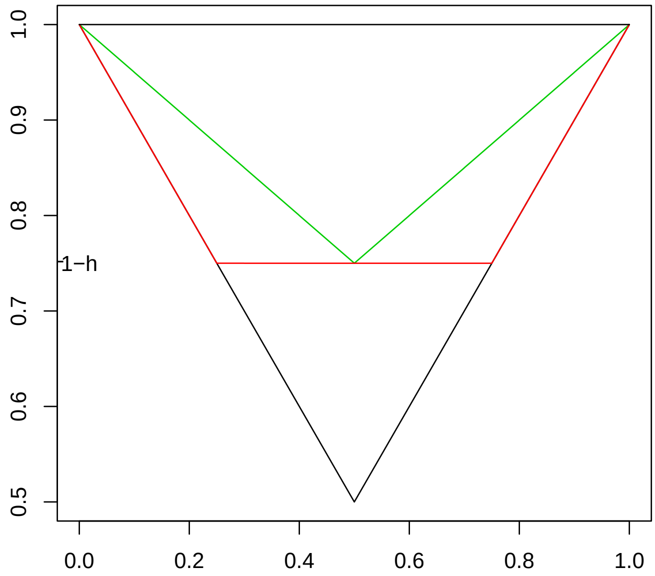
For the logistic model

$$A(\mathbf{v}) = (v_1^{1/\alpha} + v_2^{1/\alpha} + v_3^{1/\alpha})^\alpha,$$

$$A(1/3, 1/3, 1/3) = 3^{\alpha-1}.$$

α	$\tau_1 = (3 - 3^\alpha)/2$	$\tau_2 = \int_\Omega (1 - A)36/7$
0	1	1
1/4	.8420	.9457
1/2	.6340	.7670
3/4	.3602	.4559
1	0	0

For $d = 2$, how big can the difference $\tau_2 - \tau_1$ be?



Consider all (symmetric, $d = 2$) models for which $A(1/2) = 1 - h$ so that $\tau_1 = 2h$.

All the A functions must be bounded between the green graph and the red one. The green graph corresponds to a mixture model with $\alpha = 2h = \tau_1 = \tau_2$:

$$X_1 = \max(2hV_1, (1-2h)V_2)$$

$$X_2 = \max(2hV_1, (1-2h)V_3).$$

The red A corresponds to "cross over" model:

$$X_1 = \max(hV_1, (1-h)V_2)$$

$$X_2 = \max((1-h)V_1, hV_2)$$

for which

$$\tau_1 = 2h, \quad \tau_2 = 4h(1-h) = 1-(1-\tau_1)^2.$$

$$\max_h(\tau_2 - \tau_1) = \frac{1}{4}$$

$$\text{occurs at } \; h = 1/4, \;\; \tau_1 = 1/2, \;\; \tau_2 = 3/4.$$

To be fair, one could hold the area (volume) constant (i.e. τ_2) and let τ_1 vary. For instance, all triangles with height h have

$$\tau_2 = 2h, \ \ (0 \le h \le 1/2)$$

but

$$\frac{h}{1-h} \le \tau_1 \le 2h = \tau_2.$$

$$h = 1/4, \quad 1/3 \le \tau_1 \le 1/2 = \tau_2.$$

Combining two models.

$$\mathbf{X} = (X_1, \cdots, X_k), \;\; \mathbf{Y} = (Y_1, \cdots, Y_m)$$

are combined into

$$\mathbf{Z} = (X_1, \cdots, X_k, Y_1, \cdots, Y_m), \quad k + m = d.$$

To study the dependence measures of $\mathbf{Z}$ we must know the dependence between $\mathbf{X}$ and $\mathbf{Y}$. If they are independent we can compute τ_1 and τ_2 :

$$A(\mathbf{v}) = tA_1(\mathbf{u}) + (1-t)A_2(\mathbf{w}) \;\; (\mathbf{v} \in \Omega_d),$$

$$t = v_1 + \cdots + v_k, \;\; \mathbf{u} \in \Omega_k, \;\; \mathbf{w} \in \Omega_m,$$

$$u_i = \frac{v_i}{t}, \; 1 \le i \le k; \quad w_i = \frac{v_{k+i}}{(1-t)}, \; 1 \le i \le m.$$

The Jacobian of the transformation

$$(v_1, \cdots, v_{d-1}) \mapsto (t, u_1, \cdots, u_{k-1}, w_1, \cdots, w_{m-1})$$

is $$J = t^{k-1}(1-t)^{m-1}.$$

$$1 - A = t(1 - A_1) + (1-t)(1 - A_2)$$

$$S(A) = \int_{\Omega_d} (1 - A(\mathbf{v}))d\mathbf{v} =$$

$$= \int_0^1 \int_{\Omega_k} \int_{\Omega_m} (1 - A) d\mathbf{u} d\mathbf{w} t^{k-1}(1-t)^{m-1} dt$$

$$= \frac{1}{(m-1)!} \int_0^1 t^k (1-t)^{m-1} dt \int_{\Omega_k} (1 - A_1(\mathbf{u}))d\mathbf{u}$$

$$+ \frac{1}{(k-1)!} \int_0^1 t^{k-1} (1-t)^{m} dt \int_{\Omega_m} (1 - A_2(\mathbf{w}))d\mathbf{w}$$

$$= \frac{k!}{d!} S_k(A_0)\tau_{2,1} + \frac{m!}{d!} S_m(A_0)\tau_{2,2}$$

$$\boxed{\tau_2 = \tfrac{k-B_k}{d-B_d}\tau_{2,1} + \tfrac{m-B_m}{d-B_d}\tau_{2,2}}$$

where

$$B_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}.$$

For $k = m = 2$,

$$\tau_2 = \frac{6}{23}(\tau_{2,1} + \tau_{2,2}).$$

Similar, even simpler, is the treatment of τ_1:

$$A\left(\frac{1}{d}, \cdots, \frac{1}{d}\right) = \frac{k}{d} A_1 \left(\frac{1}{k}, \cdots, \frac{1}{k}\right) + \frac{m}{d} A_2 \left(\frac{1}{m}, \cdots, \frac{1}{m}\right)$$

$$= \frac{k}{d}\left(1 - \frac{k-1}{k}\tau_{1,1}\right) + \frac{m}{d}\left(1 - \frac{m-1}{m}\tau_{1,2}\right).$$

$$\boxed{\tau_1 = \tfrac{k-1}{d-1}\tau_{1,1} + \tfrac{m-1}{d-1}\tau_{1,2}}$$

For $k = m = 2$

$$\tau_1 = \frac{1}{3}\left(\tau_{1,1} + \tau_{1,2}\right)$$

Note, the sums of the weights are not equal to 1 but tend to 1 as k, m both tend to ∞.

The results here are lower bounds for τ_1, τ_2 when the two models are dependent.

Conclusions

- Conventional correlation coefficients measure pair-wise dependence, while τ_1, τ_2 are reasonable dependence measures for $d \geq 2$.

- For the mixture model τ_1, τ_2 are equal to what we desire.

- The results for combining independent models can serve as lower bounds in case the two models are dependent.

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The extremal coefficients are the natural dependence measures for multivariate extreme value distributions. For an m-variate distribution 2m distinct extremal coefficients of different orders exist; they are closely linked and therefore a complete set of 2m coefficients cannot take any arbitrary values. We give a full characterization of all the sets of extremal coefficients. To this end, we introduce a simple class of extreme value distributions that allows for a 1-1 mapping to the complete sets of extremal coefficients.Â In addition, it is shown that random variables with a multivariate extreme value distribution are associated. Applications are given to a number of parametric families of joint distributions with given marginal distributions. View. Show abstract. Extreme-value theory can be replaced in the context of the copula concept, which is encountering increasing interest in various domains of economics and finance4. Every joint distribution function can be split into its univariate marginal distribution functions and a copula function. Copulas are free of the properties of marginal distributions and present a desirable. 1 See Karolyi and Stultz (2003) for a survey. 2 See, among many others, King and Wadhwani (1990) or Baig and Goldfajn (1999).Â of two perfectly correlated variables, there is no tail dependence for the Normal multivariate distribution even for Ï â‰ 0 . Note that the extreme-value measures recently sudied by Hartmann. et al. (2004) are directly related to Ï‡ . More specifically, these authors study the probability of a. dependence structures between the extremes with two different degrees of dependence, as. measured by Kendall's tau, say. The other.Â With these choices, Cf;A belongs to the domain of attraction of the extreme value distribution with dependence function A Âˆ A?. Values of a were chosen to ensure predetermined initial degrees of dependence, namely tf;A1 Âˆ tf;A3 Âˆ 0:3 or 0.5 and tf;A2 Âˆ tf;A4 Âˆ 0:5 or 0.7. The various cases considered are summarized in Table 1. 6. Implementation of the estimators. Let Â…X1; Y1Â†; . . . ; Â… Xn; YnÂ† be a random sample from a bivariate distribution which is assumed to be in the domain of attraction of an extreme value distribution with dependence function A?.