A unified theory of gravitation and electromagnetism

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Abstract
A model of gravitation presented at the 2004 SAMS congress is applied to a closed electromagnetic system. The fine spectrum of the hydrogen atom is derived, as well as expressions for the classical electron radius and Dirac’s Large Numbers. An alternative interpretation of electron spin is also presented.

1 Introduction
In a paper presented at the 2004 SAMS conference the author showed that the Schwarzschild metric is an approximation to a generalized metric,
\[ ds^2 = e^{-R/r} dt^2 - e^{R/r} [dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2]. \]  
(1)

This metric leads to the constants of motion,
\[ E = m_0 c^2 e^{R/r}/\gamma^2, \]  
(2)
\[ L = r^2 \dot{\theta} e^{R/r}, \]  
(3)
where \( \gamma = 1/\sqrt{1 - v^2/c^2} \). These equations result in a conic for the path of a test particle,
\[ 1/r = C(1 + \epsilon \cos k\theta), \]  
(4)
which gives the observed values for perihelion precession and the bending of light.

The same results can also be derived from a Lagrangian
\[ L = -m_0 (c^2 + v^2)e^{R/r}, \]  
(5)
where \( R = 2GM/c^2 \) is the Schwarzschild radius of a massive, central body.

The gravitational redshift is found by assuming that Special Relativity (SR) is locally and instantaneously valid at all points in physical space, in particular that the time dilation formula,
\[ \gamma d\tau = dt, \]  
(6)
is valid at all points. Eqs. (2) and (6) lead to the formula,
\[ \nu = \nu_0 e^{-R/2r} \approx \nu_0(1 - R/r), \]  
(7)
which also agrees with observation.

The model distinguishes between the dynamics of the system, given by the Lagrangian, and its kinematics, which is determined by SR.
2 A closed electrodynamic system

Special relativity is a theory of electrodynamics, as attested by Einstein’s 1905 paper, *Zur Elektrodynamik bewegter Körper*. Since SR is formulated explicitly in the proposed model, it allows one to formulate the equations of electrodynamics within the gravitational model. We shall apply the results to determine the fine-structure spectrum of the hydrogen atom. Details of all calculations are given in the doctoral thesis of the author.[7]

We convert (7) to an energy equation by multiplying both sides by Planck’s constant $h$. Rearranging and applying the formula for the photo-electric effect,

$$\tilde{E} = \tilde{m}c^2 = h\nu,$$

we obtain

$$h\nu_0 = \tilde{E} e^{R/2r} = \tilde{m}c^2 e^{R/2r}.$$  

(9)

The mass $\tilde{m} = \gamma \tilde{m}_0$ is defined as the electromagnetic mass, $\tilde{m}_0$ is the electromagnetic rest mass and $\tilde{E}$ is its associated energy in SR.

Since both $h\nu_0$ and $E$ are invariant, or constant, we may relate them by a proportionality constant. As it will not affect the final equations we set this constant equal to unity. Eq.(9) can then be written as

$$E = \tilde{m}_0 c^2 \frac{e^{\Phi/c^2}}{\sqrt{1 - v^2/c^2}},$$

(10)

where we use a general potential $\Phi = Re^{2}/2r$.

We also need the Hamiltonian associated with the above equation. To find it we can show that (10) can be derived from a Lagrangian,

$$L = -\tilde{m}_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} e^{\Phi/c^2}.$$ 

(11)

From this Lagrangian we find the conjugate momenta,

$$p_r = \tilde{m}r \exp(\Phi/c^2),$$

$$p_\theta = \tilde{m}r^2 \dot{\theta} \exp(\Phi/c^2),$$

$$p_\phi = \tilde{m}r^2 \sin^2 \theta \dot{\phi} \exp(\Phi/c^2).$$  

(12)

Applying the Euler-Lagrange equations to the Lagrangian we find the following conservation equations:

$$M_z = p_\phi = \text{constant},$$ 

(13)

$$M^2 \exp(2\Phi/c^2) = p_r^2 + p_\theta^2 + p_\phi^2 = \text{constant},$$ 

(14)

where $M = (r \times \tilde{m}v)$.

The associated Hamiltonian can be written in terms of the conjugate momenta as

$$H = \left[ \tilde{m}_0 c^4 \exp(2\Phi/c^2) + c^2(p_r^2 + p_\theta^2 + p_\phi^2/\sin^2 \theta) \right]^{\frac{1}{2}}.$$  

(15)
3 The H-atom

Expanding (10) gives

\[ E = \tilde{m}_0 c^2 + \tilde{m}_0 v^2/2 + \tilde{m}_0 \Phi + \tilde{m}_0 \Phi^2/2c^2 + \tilde{m}_0 v^2 \Phi^2/4c^4 + \ldots \]  

(16)

Consider the first three terms on the right-hand side (RHS) of the above equation:

\[ E \approx \tilde{m}_0 c^2 + \tilde{m}_0 v^2/2 + \tilde{m}_0 \Phi. \]  

(17)

Compare this equation with the corresponding approximation for the gravitational model,

\[ (m_0 c^2 - E)/2 \approx m_0 v^2/2 - m_0 \Phi. \]  

(18)

The negative sign before \( m_0 \Phi \) in (18) ensures that gravitation is always an attractive force, ensuring stable, bound motion. Comparison with the Newtonian limit gave the potential \( \Phi \) as

\[ \Phi = GM/r. \]  

(19)

This simple procedure, however, cannot be followed for (17). The last two terms in this equation have the same sign, which results in unstable motion. In order for the motion of the system to be bounded, the last term must not only be inversely proportional to \( r \), but also negative. To ensure this we let

\[ \tilde{m}_0 \Phi = -e^2/r, \]  

(20)

where \( e \) is an arbitrary constant.

This is a significant result as the constant \( e \) eventually turns out to be the charge on an electron. Its existence is a mathematical requirement for bound motion. Alternately, a positive sign in (20) results in unbounded, repulsive motion.

Since \( \Phi = Re^2/2r \), (20) can also be written as

\[ \tilde{m}_0 c^2 = -e^2/r_e, \]  

(21)

where

\[ r_e = R/2. \]  

(22)

In this case \( R \) is the Schwarzschild radius of the proton.

Eq.(21) has the same form as the expression for the classical electron radius. Substituting from (20), the energy equation (10) can now be written as

\[ E = \tilde{m}_0 c^2 + \tilde{m}_0 v^2/2 - e^2/r - e^2 v^2/2rc^2 + e^4/2\tilde{m}_0 r^2 c^2 + e^4 v^2/4\tilde{m}_0 r^2 c^4 + \ldots \]  

(23)

There are two approaches to find the spectrum of the hydrogen atom, namely the method of action angle variables or a perturbation method. We shall apply both of them and show that they give the same results.
4 Action angle variables

The theory of action angle variables originated in the description of periodic motion in planetary mechanics [4, Ch. 9]. From that theory Wilson and Sommerfeld postulated the quantum condition:

For any physical system in which the coordinates are periodic functions of time, there exists a quantum condition for each coordinate. These quantum conditions are

\[ J_i = \oint p_i dq_i = n_i \hbar, \tag{24} \]

where \( q_i \) is one of the coordinates, \( p_i \) is the momentum associated with that coordinate, \( n_i \) is a quantum number which takes on integral values, and the integral is taken over one period of the coordinate \( q_i \).

Applying this rule to the conjugate momenta of (12) we find the following quantum conditions:

\[ M_z = n_\phi \hbar, \tag{25} \]
\[ M \exp(\Phi / c^2) = (n_\theta + n_\phi) \hbar \equiv k \hbar, \tag{26} \]
\[ \oint \left[ E^2 / c^2 - m_0^2 c^2 \exp(2\Phi / c^2) - k^2 \hbar^2 / r^2 \right]^{1/2} dr = n_r \hbar, \tag{27} \]

where \( n_\theta, n_\phi, k \) and \( n_r \) have the values 0, 1, 2, \ldots

To determine the atomic spectrum we need to evaluate the integral of (27), and for that we need an expression for \( \Phi \). Firstly, instead of \( R \) we use the classical electron radius \( r_e = R/2 \) as defined by (22). Also, because of the finite radius of the nucleus we choose an arbitrary effective nuclear radius of \( gr_r \). The potential term is then written as

\[ \exp\left(\frac{2\Phi}{c^2}\right) = \exp\left(\frac{2r_e}{r - gr_e}\right), \tag{28} \]

so that

\[ \exp(2\Phi / c^2) = 1 + 2 \frac{r_e}{r} + 2 \frac{r_e^2}{r^2} (g + 1) + 3 \frac{r_e^3}{r^3} g (g + 1) + \ldots \tag{29} \]

For convenience we also define a parameter \( f \) such that

\[ f = 2(g + 1). \tag{30} \]

We shall subsequently see that the value of \( g \), or \( f \), is related to the concept of electron spin.

Approximating (29) to second order in \( r_e / r \), substituting this approximation in (27) and integrating gives

\[ E_m^2 = 1 - \frac{\alpha^2}{\left[ n - k + \sqrt{k^2 + f \alpha^2}\right]^2}, \tag{31} \]
where \( E_m = E/\bar{m}_0c^2, n = n_r + k \) and \( \alpha = e^2/\hbar c \) is the fine structure constant. This expression is simplified by expanding to fourth order in \( \alpha \):

\[
E_m \cong 1 - \frac{\alpha^2}{2n^2} \left[ 1 + \frac{\alpha^2}{n} \left( \frac{1}{4} - \frac{f}{k} \right) \right]. \tag{32}
\]

The corresponding Sommerfeld/Dirac expressions are respectively

\[
E_m^2 = \left( 1 + \frac{\alpha^2}{n - k + \sqrt{k^2 - \alpha^2}} \right)^{-1} \tag{33}
\]

and

\[
E_m \cong 1 - \frac{\alpha^2}{2n^2} \left[ 1 + \frac{\alpha^2}{n} \left( \frac{1}{k} - \frac{3}{4n} \right) \right], \tag{34}
\]

where \( k = j + 1/2 \) for the Dirac expression, \( j = 1/2, 3/2, 5/2, \ldots (n - 1)/2 \). The difference between the energy given by our model \( E_W \), as given by (32), and that of the Sommerfeld-Dirac model, \( E_D \), as given by (34), is

\[
(E_D - E_W)/\bar{m}_0c^2 = \frac{\alpha^4}{2n^3} \left[ \frac{1}{k}(f + 1) - \frac{1}{n} \right]. \tag{35}
\]

It will be shown in the next section that this difference corresponds to the energy associated with the "spin-orbit" interaction of our model.

### 5 Perturbation method

We use this method as applied by Born and others [1, Ch.4]. In order to compare some of the results with classical electrodynamics we shall need to refer to the classical Hamiltonian for an electron in an electromagnetic field. This is given by [6, p124]:

\[
H_{\text{classical}} = e\Phi + \left( \frac{p}{c} - eA \right)^2 / 2\bar{m}, \tag{36}
\]

where \( \Phi \) is the electrostatic potential and \( A \) is the vector potential. Substituting the appropriate expressions for \( \Phi \) and \( A \) gives [7, 8.3.1]

\[
E_{\text{classical}} = -\frac{e^2}{r} + \frac{p^2}{2\bar{m}} - \frac{e^2v^2}{2c^2r} + \frac{e^4v^2}{8}\bar{m}c^4r^2. \tag{37}
\]

To apply the perturbation method we need to express the energy \( \tilde{E} \) in terms of the momentum:

\[
E = (p^2c^2 + \bar{m}_0c^4)^{1/2} \exp(\Phi/c^2). \tag{38}
\]

Again, taking the finite radius of the nucleus into account, we choose for the potential,

\[
\exp(\Phi/c^2) = \exp(r_e/(r - gr_e)), \tag{39}
\]

\[5\]
so that the potential term can be written as
\[
\exp(\Phi/c^2) = 1 + \frac{r_e}{r} + w\frac{r_e^2}{r^2} + (w^2 - 1/4)\frac{r_e^3}{r^3} + \ldots, \tag{40}
\]
where
\[
w = (g + 1/2) = (f - 1)/2. \tag{41}
\]
With this form for the potential, and using \(\tilde{m}_0 c^2 r_e = -e^2\), (38) can be expanded as
\[
E = \tilde{m}_0 c^2 \left[ \frac{p^2}{2\tilde{m}_0} - \frac{e^2}{r} - \frac{p^4}{8\tilde{m}_0^3 c^2} + \frac{p^2 r_e}{2\tilde{m}_0 r} + w\frac{r_e^2 \tilde{m}_0 c^2}{r^2} \right]
\]
\[+ \frac{w^2 r_e^2}{2\tilde{m}_0 r^2} \left[ \frac{p^4 r_e}{8\tilde{m}_0^2 c^2 r} + \tilde{m}_0 c^2 (w^2 - 1/4)\frac{r_e^3}{r^3} + \ldots \right]. \tag{42}
\]

Applying the unperturbed Bohr theory to each braced term, we find the following quantized expressions:

**5.1 \(E_0\): Rest mass energy**

The first term on the right is the rest mass energy, which we denote by \(E_0\):
\[
E_0 = \tilde{m}_0 c^2. \tag{43}
\]

**5.2 \(E_1\): Bohr energy**

The next two terms represent the unperturbed Coulomb energy of the hydrogen atom, which we indicate by \(E_1\):
\[
E_1 = \frac{p^2}{2\tilde{m}_0} - \frac{e^2}{r}. \tag{44}
\]

According to the Bohr theory,
\[
E_1 = -R_e/n^2, \quad n = 1, 2, \ldots \tag{45}
\]
where
\[
R_e = R_y \frac{hc}{e^2/2a_0} = \alpha^2 \tilde{m}_0 c^2/2
\]
\[
a_0 = \text{Bohr radius} = \frac{\hbar^2}{\tilde{m}_0 e^2}
\]
\[
R_y = \text{Rydberg constant} = 2\pi^2 e^4 \tilde{m}_0 c/\hbar^3 = \alpha/4\pi a_0. \tag{46}
\]

**5.3 \(E_2\): Relativistic correction**

The third term is denoted by \(E_2\). It can be shown[7, Appendix I] that
\[
E_2 = -\frac{p^4}{8\tilde{m}_0^3 c^2}, \tag{47}
\]
\[= -\frac{\alpha^2 R_e}{n^3} \left[ \frac{1}{k} - \frac{3}{4n} \right]. \tag{48}
\]
This is the “relativistic correction” of the Bohr-Sommerfeld model [1, par33]. This energy term is similar to that contained in the Dirac expression of (34). The sum of $E_0$, $E_1$ and $E_2$ gives an expression identical to that of Sommerfeld and similar to that of Dirac.

The fact that the Wilson-Sommerfeld and Dirac models give the same values for the fine structure of the hydrogen spectrum has been described as “one of the most amazing coincidences to be found in the study of physics”[2, p356].

It is well-known that Sommerfeld’s result was fortuitous as the effect of spin-orbit coupling was ignored in his model. This effect is incorporated in the Dirac model. In our model we shall see below that $E_3$ is an orbit-interaction term and that $E_4$ is related to ‘electron spin’. These two terms, missing in the Sommerfeld model, can now be added to the Sommerfeld energy expression given by $E_0 + E_1 + E_2$.

5.4 $E_3$: Orbital magnetic energy

We denote the fourth term by $E_3$:

$$E_3 = p^2 r_e / 2\tilde{m}_0 r.$$  \hspace{1cm} (49)

We use the results of the unperturbed Bohr theory to evaluate this energy. [7, Appendix I]

From (44),

$$E_3 = (E_1 + e^2 / r) r_e / r$$

$$= r_e (E_1 / r + e^2 / r^2).$$  \hspace{1cm} (50)

Using (45) and the average values [1, p144],

$$1/r = 1/n^2 a_0,$$  \hspace{1cm} (51)

$$1/r^2 = 1/a_0^2 n^3 k, \quad k = 1, 2, \ldots n$$  \hspace{1cm} (52)

and

$$r_e / a_0 = a^2,$$  \hspace{1cm} (53)

we get

$$E_3 = \alpha^2 R_e \left( \frac{2}{k} - \frac{1}{n} \right) = \frac{\alpha^4 \tilde{m}_0 c^2}{2n^3} \left( \frac{2}{k} - \frac{1}{n} \right)$$  \hspace{1cm} (54)

The physical interpretation of $E_3$ is that it is the energy due to the magnetic interaction of an electron moving in orbit about a proton. This can be seen as follows.

Substituting $p = \tilde{m} v$ and $r_e = -e^2 / \tilde{m}_0 c^2$ into (49) gives
\[ E_3 = -\frac{e^2 v^2}{2rc^2} \left( \frac{\tilde{m}}{\tilde{m}_0} \right)^2, \]
\[ \approx -\frac{e^2 v^2}{2rc^2} \quad \text{in the non-relativistic limit.} \quad (55) \]

It corresponds to the third RHS term of (37), i.e. the magnetic energy due to orbital motion \[ \text{[7, Appendix I]}: \]

\[ E_{\text{magn}} = \mu_\ell \cdot H, \]
\[ = \frac{g_\ell \mu_B}{\hbar} \mathbf{M} \cdot \mathbf{H}, \]
\[ = -\frac{e^2 v^2}{2rc^2} \quad \text{in the non-relativistic limit.} \quad (56) \]

\textbf{5.5 \( E_4 \): ‘Electron spin’}

\[ E_4 = \frac{e\hbar^2}{m_0 c^2} / r^2, \]
\[ = we^4 / \tilde{m}_0 c^2 r^2. \quad (57) \]

Applying (52) gives

\[ E_4 = w2\alpha^2 R_e / n^3 k = w\alpha^4 \tilde{m}_0 c^2 \frac{1}{n^3 k}. \quad (58) \]

We consider the significance of the factor \( w \).

We note that the potential energy expression (40) can be truncated after the quadratic term in \( r_e / r \) by letting \( w^2 - 1/4 = 0 \). As such, truncation can be considered as the limit to the resolution of the apparatus used for spectral observation. With this condition, we find that

\[ w = \pm \frac{1}{2}. \quad (59) \]

gives the spectrum due to all interactions up to second degree in \( r/r_e \).

Therefore, from (46) and (58):

\[ E_4 = \pm \frac{1}{2} \frac{e^8 \tilde{m}_0}{\hbar^4 c^2 n^3 k}. \quad (60) \]

The above expression for \( E_4 \) corresponds to the quantum mechanical result for the energy due to electron spin. Except for the quantum numbers, Eisberg and Resnick [3, Example 8-3] find a similar result for the energy due to spin-orbit interaction.

The equivalence of (60) to the result of Eisberg and Resnick also confirms the implicit value \( g_s = 2 \) in \( E_4 \).

In this study \( E_4 \) corresponds to the energy due to quantum mechanical spin only. Combining \( E_3 \) and \( E_4 \) gives the corresponding total spin-orbit energy.

For \( k = 1 \) the expression for \( E_4 \) is equal to the Darwin term of the Dirac theory. In the Dirac theory the Darwin term has to be introduced separately for \( \ell = 0 \) states, whereas in our model \( E_4 \) already provides for \( \ell = 0 \) through the degeneracy \( (\ell = 0, 1) \) associated with the \( k = 1 \) level.
5.6 \( E_5 \): Radiative correction

\[
E_5 = w \frac{p^2 r_e^2}{2m_0 r^2},
\]

\[
= \pm \frac{1}{2} \alpha^4 R_e \left[ \frac{1}{n^5 k} - \frac{2}{n^3 k^3} \right].
\]  

(61)

Substituting \( p = \tilde{m} v \) in \( E_5 \) gives

\[
E_5 = \pm \frac{1}{2} v^2 \epsilon^4 \left( \frac{\tilde{m}}{\tilde{m}_0} \right)^2.
\]  

(62)

In the non-relativistic limit, \( \tilde{m} \approx \tilde{m}_0 \), the above term corresponds to the last RHS term of (37), i.e. the classical energy resulting from radiative reaction. Its value is too small (\( \sim 10^{-7} \) ev) to affect the values of the fine-spectrum.

5.7 Summary

Each term in (42) can be related to a standard electrodynamic effect. It is significant that although (42) does not explicitly contain any vector quantities, such as the vector potential \( A \), this potential is implicit, as shown in the discussion of \( E_3 \) and the comparisons with (37).

An explanation for the difference (35) between the spectrum of the proposed model and that of Dirac-Sommerfeld can be seen as follows:

Consider the sum:

\[
E_3 + E_4 = \frac{\alpha^2 R_e}{n^3} \left[ \frac{2}{k (w + 1)} - \frac{1}{n} \right]
\]  

(63)

or, since \( w = (f - 1)/2 \),

\[
E_3 + E_4 = \frac{\alpha^2 R_e}{n^3} \left[ \frac{1}{k (f + 1)} - \frac{1}{n} \right].
\]  

(64)

The above equation corresponds to (35), the difference between the Sommerfeld-Dirac expression and that of this paper. The expression (32) therefore already incorporates the spin-orbit interaction.

6 The Large Number Coincidences

Dirac postulated that the large dimensionless ratios (\( \sim 10^{40} \)) of certain universal constants underlie a fundamental relationship between them. A theoretical explanation for these ratios have not yet been found, but it became known as Dirac’s Large Number Hypothesis (LNH)[5]. Some of these relations are derivable from (22).
Taking $R$ as the Schwarzschild radius of the proton, $R_p = \frac{2GM_p}{c^2}$, we rewrite (22) as

\[-\frac{e^2}{\tilde{m}_0 c^2} = \frac{GM_p}{c^2}\]

or

\[-\frac{e^2}{GM_p \tilde{m}_0} = 1. \quad (65)\]

Defining the relationship between the gravitational mass $M_p$ and the electromagnetic rest mass $\tilde{m}_0p$ of the proton as

\[M_p = N \tilde{m}_0p, \quad (66)\]

where $N$ is a dimensionless constant, we can write (65) as

\[-\frac{e^2}{GM_0p \tilde{m}_0} = N, \quad (67)\]

which, if the absolute value is taken, is the basic relationship of the LNH.

7 Lorentz force

In a unified theory one expects that both gravitation and electromagnetism be described by a single formulation. From the respective Lagrangians of (5) for gravitation, and (11) for electromagnetism, we find a single form for the Lorentz force for both interactions [7, s8.4]:

\[\dot{p} = \tilde{m}[kE + v \times H],\]

Gravitation : $k = -1$,

Electromagnetism : $k = 1$,

where

\[E = \frac{GM}{r^2} = \frac{r_e c^2}{r^2}, \quad (69)\]

\[H = \frac{GM(v \times r)}{c^2 r^3} = \frac{r_e v \times r}{r^3}. \quad (70)\]

It must be emphasized that the force equation (68) satisfies both the classical tests for gravitation as well as providing a description of the electrodynamics of a closed system.

8 Conclusion

The proposed model relies on a distinction between the dynamics and kinematics of a system. The fundamental force is that of gravitation, given by the Lagrangian of (5). This interaction gives rise to the motion of a particle and this motion, or its kinematics, is described by SR. This in turn manifests as the electrodynamics of the system. This interaction is described by the Lagrangian of (11), which is derived from redshift formula of gravitation.

The form of the equation for the Lorentz force (68) confirms the unifying relationship between the two interactions.
References


The goal of this theory is finding relation between all known fields like gravity, electromagnetism etc. Let's start from something small, for example, find relation between gravitational constant $G$ and electromagnetic constants $\varepsilon_0$ (Vacuum permittivity), $\mu_0$ (Vacuum permeability) and some other constants like $\pi$ number, Planck's constant, Elementary charge (electron charge), Boltzmann constant. Here full random brute force method is suggested in order to find this relation. Theory of probability will be used to cover all possible combinations and it is explained later. Well from units point of view it should not be here, because only this constant has Kelvin, but who knows, maybe we will find something very new here, like relation between thermodynamics and gravitation.