Geometry, Relativity, Geodesy

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This book, edited by Wihmann in 1993 and never re-edited, remains nowadays as a valuable and significant reading for anyone interested in understanding, addressing and implementing the complex field of relativity linked to the Space Geodesy techniques. Over the years, the increasing accuracy measurement has led to the necessity of taking into account the effects due to relativity. An example is the Global Navigation Satellite System (GNSS). This technique, intended to achieve a more precise positioning via triangulation, requires exact synchronization between satellite clocks (atomic clocks) and watches located on the Earth's surface. GNSS allows better positioning systems and affects positively a great variety of socioeconomic and technical activities.

According to the Einstein’s theory of relativity, the space-time curvature is a direct result of the presence of matter in the Universe. Consequently, a clock located on a satellite at a given point in space, will be affected by this theory. While its movement will make the time goes slower than a fixed clock on the Earth, farther from the Earth's gravitational field will experience further acceleration[1]. Another example is the very long baseline interferometry (VLBI), based on the simultaneous observation by two or more terrestrial radio telescopes to celestial objects in order to determine geodetic parameters through an inverse analysis technique. Observed objects with VLBI (quasars) have the distinct feature of being very bright although they are very distant, placing them at distances of hundreds or even thousands of millions of light years. As a result, the radio signal received on Earth is affected by cosmological phenomena such as gravitational lensing or gravitational waves which are based on the theory of relativity.

The study of relativity basically implies the knowledge of differential geometry and tensor calculus. At present there are a lot of books aimed at teaching these fields. However, prior knowledge and appropriate management of the notation is often necessary. The book "Geometry, Relativity, Geodesy", although not intended to be a book merely about relativity, covers three main aspects:

[1] Time runs differently for a clock located in a satellite and another placed on Earth. The clock is moving in orbit and has a positive speed relative to the Earth clock. The latter is immersed in the Earth's gravitational field while the satellite is much further away and gravity intensity for it is lower. According to the theory of relativity, the moving clock experiments a time dilation (ie , a delay) in relation to the other clock, while at the same time this will contract (ie , an advance) time with a lower gravitational field. What slow down time is the speed and gravity.

- Representation of the indexed notation or Einstein's notation, essential to preserve formulas regardless of the dimension.
- Development of mathematics necessary to establish the foundations of special and general theory of relativity.
- Connection between geodesy and relativity using existing applications.

This is possible due to the background of the authors and extensive research in these two fields of science. Helmut Moritz has focused on the study of Physical Geodesy, relativity works since 1967, especially in kinematic geodesy. Moreover, Bernhard Hofmann-Wellenhof, Geometric Geodesy Professor since 1986, works in the topic of relativity since 1990 in connection with GNSS.

Initially the authors compare Einstein summation convention against the matrix notation, the main advantage of the first one is that it works with scalars instead of matrices. It implies rejecting the checksums to allow a clearer formulation based on indices. Throughout the book concepts are defined sequentially in terms of space and the coordinate system previously defined. They start from the simplest dimensional case and end with the n-dimensional general case. This sequence allows the reader for having an insightful transition from the special to the general relativity.

The first chapter focuses on linear geometry, ie, the Euclidean spaces with rectangular (or Cartesian) coordinates and oblique (or affine) coordinates. The concept of metric tensor as well as the operations of each system, which involve the introduction of covariant and contravariant vector for the system of affine coordinates, have been developed. Finally the n-dimensional case is generalized to prove the invariance of formulas regardless the dimension.

The second chapter focuses on the differential geometry. The nonlinearity is introduced with curvilinear coordinates in two-dimensional Euclidean space. The line item (or first fundamental form) and the expression of the associated metric tensor are presented introducing the Gaussian fundamental quantities. The equation of a straight line is developed. This equation indicates the trajectory of a point in a gravitationally curved space-time within the framework of general relativity. From the equation the authors show how to derive the Christoffel symbols of first and second kind. A new definition of the derivative, covariant derivative (or absolute), is introduced as basis vectors are not constant in curvilinear coordinates. They conclude that the covariant differential of a scalar may be defined equal to the ordinary differential, with a constant metric tensor in the sense of covariant differentiation.

Following, the surface vectors are defined to derive minimum connection between two points or geodesics that satisfy the equation of a straight line in n-dimensional curved spaces (or Riemann spaces). The direct and inverse problems of geodesy are stated and the solution is expressed in terms of the Gaussian fundamental quantities making used of the general Legendre series. The above problems, only related to orthogonal coordinates, lead to the generalization of the geodesic equation in non-orthogonal coordinates. It allows obtaining the non-orthogonal curvilinear coordinates or Riemannian coordinates. The existence of the first fundamental form for any n-dimensional space leads to the Riemann n-dimensional space (or Riemannian manifold). Everything related to the metric tensor belongs to the intrinsic geometry of a surface and has the advantage of being independent of the coordinate system and the dimension.[2]

In the final part of this chapter the authors explore the concept of intrinsic and extrinsic curvature (with respect to integral space) to obtain a measure of the curvature that only depends on the intrinsic properties of the surface. Gaussian curvature is defined in terms of the metric tensor and the Riemann curvature tensor. Finally the equation of the eiconal, exceptionally included in a book about Riemannian or differential geometry, is developed and it plays a fundamental role in mechanical and geometrical optics.

In this chapter the generality of the first fundamental power extended to any coordinate system and any dimension is concluded. The expression in four dimensions is the one needed for general relativity purposes.

The third chapter introduces the nonlinear coordinates in three-dimensional Euclidean spaces. Local Cartesian and natural coordinates are defined. Thus, the concepts applied to curvilinear coordinates in the plane are generalized to three-dimensional spaces to derive Marussi tensor. This tensor, defined from Eötvös tensor (or gradient tensor), expresses the dynamic and geometric properties of the Earth's gravitational field making use of the second derivatives of the gravity potential. The Bruns equation, Pizzetti's theorem and the metric tensor are developed in terms of the elements of the Marussi tensor previously described. These relationships are shown to describe the role of intrinsic geodesy, i.e., that one that locally describes the Earth's gravitational field using only quantities that have a physical meaning and therefore are observable.

Next the authors explore the path of a ray of light according to the theory of refraction, and the application of the eiconal equation. Finally the curvature of space refraction is proved. The aim of this chapter is to show the practical application of mathematical formulas defined in three dimensional spaces, as well as to distinguish the types of conformal transformations between spaces of two and three dimensions.

The fourth chapter deals with the theory of relativity starting from the special relativity. It starts introducing classical mechanics for the relative motion between two inertial reference systems to derive step by step the Lorentz transformations that are applicable to velocities near the speed of light[3]. The metric introduced in previous chapters is now used to define the associated special relativity and the four-dimensional space-time. The authors digress from Euclidean metric to derive a pseudo-Euclidean. The line element is obtained by extending the three-dimensional case for a period of time, and they show its invariance regarding

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[2] When we refer to the intrinsic geometry of a surface we mean the properties that can be measured within it without resorting to the space that contains it. This is not so the case for the extrinsic geometry where studying properties requires information from the space that contains it.

[3] Lorentz transformations are used to link two inertial reference systems (not rotating), such as the relative velocity relative to each other is near that of light. Classical mechanics is valid for systems whose velocity is very small, but fails at high speed, so that for very small or 'ordinary' speed the Lorentz transformation is equivalent to the Galilean one.

Lorentz transformation. It continues in a logical manner with the space-time diagram for a light ray that generates a light cone, where the events of the past, present and future are represented.

The geodesic defined above, is again introduced to represent the path followed by a material particle in the cone of light. The different types of geodesics are explained and multiple graphics and images are presented to help the reader to clarify the abstract aspect of the theory. Following, several concepts and applications resulting from the unification of space and time are explained: Proper time and time dilatation, the Doppler Effect and the aberration of light. The first part of the chapter ends with a brief introduction to relativistic kinematics by introducing the concepts of velocity and acceleration in the four-dimensional space.

The second section focuses on general relativity. The expression of the line element in a noninertial system is obtained from the inertial system of special relativity thus obtaining the metric in the non-inertial system. Next different expressions of the metric or metric tensor are derived when the system is also in a gravitational field, or when the system is not rotating but there is gravity. They continue redefining the trajectory of a particle in a gravitational field, due to the curvature of space-time. Then the authors include an analogy between the situation in space-time and the two-dimensional space to summarize the concepts previously presented. The usefulness of the Riemann curvature tensor is fully explained and Einstein's law of gravitation is introduced. They prove Newton's theory of gravitation as the non-relativistic classical limit of Einstein's theory, i.e., for weak gravitational fields and slow motions. Finally the Schwarzschild metric, one of the solutions of the equations of Einstein's gravitational field, is developed.

The fifth chapter focuses on the geodesic applications of relativity. It begins with the presentation of the phenomena and the techniques where relativity is present. Then the effects on geodesic measurements are stated. The treatment in previous chapters of gravitation and inertia independently allows the introduction of the inertial surveying, aerial gravimetry and gradiometry in this chapter as there is a direct involvement of the concepts previously presented.

The sixth and final chapter is devoted to Hilbert spaces. This theory, the mathematical basis of general relativity, is crucial for making used of the least squares in the geodesic context and for the formulation of quantum mechanics. The purpose of this chapter is to lead prior results to the case of an infinite number of dimensions.

This book is characterized by a sequential development, with multiple references, examples and detailed exposition of mathematical results. Accordingly we have a worthwhile work that after twenty years remains a keystone for geodesy, introducing in an outstanding manner key concepts widely used nowadays. Given the difficulty to find copies of this work, a new edition is necessary and highly recommended.

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Geodesy Teaching Language Suomi Course Sta and Contact Info Martin Vermeer, room Technopolis Innopoli 3D (Vaisalantie 8). 4.143, martin dot vermeer at aalto dot. i.Â The formulas of spherical geometry are very useful in geodesy. The surface of the Earth, which to rst approximation is a plane, is in second approximation (i.e., in a small, but not so small, area) a spherical surface. Even in case of the whole Earth, the deviation from spherical shape is only 0.3%. In general relativity, a geodesic generalizes the notion of a "straight line" to curved spacetime. Importantly, the world line of a particle free from all external, non-gravitational forces is a particular type of geodesic. In other words, a freely moving or falling particle always moves along a geodesic. In general relativity, gravity can be regarded as not a force but a consequence of a curved spacetime geometry where the source of curvature is the stressâ€"energy tensor