The discovery of E.W. Beth's semantics for intuitionistic logic

A.S. Troelstra and P. van Ulsen

Abstract

One of van Benthem's predecessors, and the first in line of the Dutch logicians, was Evert Willem Beth. During the last decade of his life he worked on a truly constructive semantics for intuitionistic logic, with a corresponding completeness theorem. The result is known as "Beth models". We try to describe his intents and efforts, but it is not possible to give a clear story line: the data are too scarce. However, we attempt a reconstruction. For this we not only used published records, but as much as possible also quotes from correspondence. Beth semantics combine trees, tableaus, choice sequences and fans. Intuitionistically acceptable completeness required Beth to avoid certain classical notions, in particular König's Lemma. Instead, Beth used Brouwer's fan theorem.

Contents

1 Introduction

Johan van Benthem occupies a position which was once held by Evert Willem Beth. Johan's 50th birthday seemed a suitable occasion for investigating the work of one of his scientific ancestors; so here we take a look at Beth's discovery of "Beth semantics" and completeness for "Beth models".

To our regret, we do not have a clear story line: the data are too scarce. But we have done our best, and have sketched a reconstruction. For this reconstruction, we used not only published records, but as much as possible also quotes from correspondence. The corrrespondence is preserved at Haarlem (Rijksarchief in Noord-Holland) with the name of "E.W. Beth Archive".

2 Semantics for intuitionistic logic "before Beth"

In L.E.J. Brouwer's writings from 1908 onwards there is, more or less implicitly, an informal intuitionistic interpretation of certain logical operators such as "or", "implies", and "not".

This interpretation was formulated more precisely, as an informal interpretation of formal logical expressions by A. Heyting, first in Heyting (1931), and later, more completely in the monograph Heyting (1934). A similar interpretation was formulated by A. Kolmogorov in a paper which appeared in 1932. Originally Heyting and Kolmogorov regarded their respective interpretations as distinct, but later Heyting came to regard them as essentially the same. It is for this reason that this interpretation is usually referred to as the Brouwer-Heyting-Kolmogorov interpretation, or BHK-interpretation.

Where classical semantics describes how the truth-value of a logically compound statement is determined by the truth-values of its components, the BHK-interpretation describes what it means to *prove* a logically compound statement in terms of what it means to *prove* the components. In this explanation, "constructive proof", "constructive method" appear as primitive notions. Kolmogorov interpreted statements as problems, and the solving of the problem associated with a logically compound statement is explained in terms of what it means to solve the problems represented by the components. The connection with Heyting's formulation becomes clear if we think of "solving the problem associated with statement A" as "proving A".

Because of the vagueness of the notions of "constructive proof", "constructive operation", the BHK-interpretation has never become a versatile technical tool in the way classical semantics has. Perhaps it is correct to say that by most people the BHK-interpretation has never been seen as an intuitionistic counterpart to classical semantics.

Another approach to semantics for intuitionistic logic started with Jaskowski (1936) who used certain systems of truth-values, special cases of Heyting algebras, for a completeness proof for intuitionistic propositional logic.

Stone (1937) does not explicitly state a completeness theorem, but in fact his paper contains a completeness result for intuitionistic propositional logic. Tarski (1938) obtains a completeness result for intuitionistic propositional calculus IPC w.r.t. valuations using open sets in topological spaces (in particular $\mathbb{R}^n$) as values. Tarski also sketches an extension to valuations in a suitable class of lattices (More about this in Troelstra & van Dalen (1988), p. 731). The Kripke-tree semantics Beth turned to in the final years of his life belongs in this category too. It seems to have no direct relation to the meaningful semantics discussed in the present article (see de Jongh & van Ulsen (1998)).

Finally, in 1945 S.C. Kleene published an interpretation (realizability) of formalized intuitionistic arithmetic in the structure of the natural numbers. This was truly a semantics, linking formal expressions to truth in a familiar structure.

It is against this background that Beth's interest in an intuitionistically meaningful semantics for intuitionistic predicate logic arose. But Beth was not himself an intuitionist as it appears in a letter to K. Schröter of July 16, 1960:

> In your letter you call me a "follower of the intuitionistic foundations of mathematics". Although I am strongly interested in intuitionism, and know this point of view for geographical reasons very well, more than other workers in the field, I do not want to be considered as a follower, let alone as an exponent. I would describe my point of view as tolerant-classical.

3 Beth's Indagationes paper from 1947 and the Berkeley lecture in 1952

According to Beth's own statement in Beth (1959*b*), p. 15, his concern with the semantics of intuitionistic logic dates from 1943. In 1946 Beth was appointed in Amsterdam as the first Dutch professor in logic. Already in 1947 Beth published a paper in the Proceedings of the Royal Dutch Academy of Sciences with the title "Semantic considerations on intuitionistic mathematics."

The discussion in this paper is of a very general nature; Beth explains at some length the idea of semantics in Tarski's sense. He then turns to intuitionism; as its key notion he sees the notion of a spread. He cites Brouwer's definition and then recasts it in a more formal way. Later Heyting, in his book "Intuitionism, an introduction" gave a simplified and clear presentation of Brouwer's notion. Beth's reformulation may be seen as a first step in this direction. (This point was appreciated by S.C. Kleene in his review of Beth's paper in the Journal of Symbolic Logic in 1948.) Beth also asks whether the spread law could always to be taken to be recursive; he thought not, from an intuitionistic point of view. Recursiveness was to narrow for intuitionistic purposes. Beth got his credits, together with Kleene, in Odifreddi (1989), p. 118. But on the whole the content of this paper is quite meager, although it should not be forgotten that Tarski's notion of semantics dating from 1935 was at that time still relatively unfamiliar to many.

There exists a draft in Dutch of this paper, which apparently was sent together with a letter to L.E.J. Brouwer; the letter is dated july 7, 1945. Curiously enough, the letter starts with "Herewith I take leave to send you some remarks on non-intuitionistic mathematics [. . .]", while the manuscript which is supposed to have been sent with this letter almost exclusively deals with intuitionism. So is this a typing error, or do the manuscript and the letter not really belong together? Perhaps neither of these: it probably was a joke, typical for Beth's sense of humor, according Dick de Jongh.

Beth returned to spreads, and in particular Brouwer's "fan theorem" for finitary spreads in the lecture "Compactness proofs in intuitionistic mathematics" at the Mathematics Colloquium at Berkeley (May 15, 1952; ms. in two versions). In this lecture Beth presents a version of Brouwer's proof of the bar theorem and the fan theorem; Beth states that the version is due to a student of Heyting, Pinxter. In the argument as presented by Beth, some subtleties of Brouwer's original argument have been swept under the carpet. Beth then proceeds to show that the fan theorem can fulfill the role of classical compactness in intuitonistic mathematics; he discusses in particular how the Heine-Borel covering theorem for the interval may be obtained from the fan theorem. He realizes that the application of the fan theorem is not entirely straightforward, but that one must choose the right sort of spread representation for the interval (a fact known to Brouwer). In short, the paper does not contain any new result, but it testifies to Beth's ongoing preoccupation with Brouwer's proof of the fan theorem and shows his awareness of the fact that if one wants to find an intuitionistic substitute for a classical compactness argument, one needs the fan theorem. It is remarkable that Beth never — not in this lecture but also not in his later work — referred to Brouwer (1926). There Brouwer presented an intuitionistically acceptable variant of Heine-Borel.

4 On the Paris lecture of 1955 and the tools for constructive semantics

Beth made real progress with the semantics of intuitionistic predicate logic only after he had invented the semantic tableaus for classical logic in 1954 (later published as Beth (1956*a*)). In Beth (1955*a*), the next step, he tried to formulate a decision method for intuitionistically acceptable formulas, given that they are classically acceptable. He used semantic tableaux, and their deductive tranformation. Beth (1955*b*), p. 341–342: "I stated the conjecture that semantic tableaux might provide us a decision procedure for intuitionistic validity of classically valid formulas." Beth's hope was destroyed by Kleene's letter of June 7, 1955 (Madison). Kleene gave some counterexamples and finished his objections as follows:

> There can be no decision procedure for the problem 'let A be a formula which holds classically; does A also hold intuitionistically?' [...] At any rate I do not see how to rule out such possibilities [i.e. Kleene's counterexamples]; and so it does not seem implausible that increasing the number of individuals taken into consideration [as Beth proposed] might give room for a correct proof intuitionistically in place of one correct classically but incorrect intuitionistically.

For Beth's agreement with Kleene see Beth (1955*b*), p. 341–342.

The intuitionistic system G3 in Kleene's "Introduction to Metamathematics" (p. 481), was taken by Beth as the basis for his syntax. Out of this Beth developed his tableaux (trees with attached formulas); or, as Beth said in his "Foundations of mathematics", p. 450,

> If read upside down, the above [i.e. intuitionistic syntactical] rules may also be construed as instructions for the construction of a semantic tableau.

Kleene, and his predecessor G. Gentzen, had disjunctive and conjunctive clauses (for the tableaux the conjunctive and disjunctieve splittings). Beth added infinite regression for the branches of his trees; this was produced by cyclic permutation of formulas.

It is not very clear how Beth discovered his semantics. Correspondence about this topic is of a later date and Beth never gave a precise description of the way he arrived at his ideas. So we use a lot of quotations from sources later than 1956, but a real reconstruction is impossible.

On May 3, 1957 Beth sent a letter to Heyting from Baltimore, while he was a visiting professor at Johns Hopkins University. In that letter, written after his discovery, you can find a clue to the beginnings of the intuitionistic use of semantic tableaux:

> Indeed I start from a classical, hence intuitionistically debatable notion of model; however, my considerations concern classical logic. Moreover, a considerable part of the construction is nevertheless intuitionistically acceptable, since the tree construction associated with a logical problem produces a collection of models which can be represented by a finitary spread.

In the same letter Beth told that he was examined by G. Kreisel:

> Concerning my construction of intuitionistic logic, I had in Princeton an interesting conversation with Kreisel, who seemed rather taken with the idea; he proposed an exam-question, which I was able to answer satisfactorily.

In 1955 Beth gave a lecture in Paris on the semantics of intuitionistic logic, in which he sketches the essentials of his semantics. The lecture was published in 1958, only *after* his more substantial paper on the same topic had appeared in 1956.

The lecture of 1955 is a big step forward. Beth sketches the clauses for intuitionistic validity in his models, and discusses some examples. However, there is still no indication of a completeness proof, and one of the clauses for validity is manifestly inadequate (namely, the clause for the validity of an existential statement). The absence of a completeness proof is not so strange, if you read Beth's letter of September 5, 1955 to A. Robinson (italics by the authors):

> In my paper on intuitionistic logic, I take a non-intuitionistic attitude, and convince myself that, if an intuitionist uses a law of logic not contained in Heyting's system, then I can find an intuitionistic counter-example to prove that this law is not acceptable from an intuitionist point of view. For the denumerable case, the lemma of infinity is not needed, we only need the (metamathematical) principle of the excluded third. *As for the classical logic, an entirely constructive completeness proof cannot be given.* But the counter-example, the existence of which is proved by non-constructive methods, is in itself constructive.

Within a year Beth changed his mind. His next step, in 1956, was the development of semantics with constructive completeness for intuitionistic logic. For his formulation of validity Beth made use of trees because of their relationship with tableaux. In Beth (1956*b*), p. 379, he motivates his use of choice sequences as follows (italics by Beth):

> [...] the introduction of the notion of a choice sequence brings a subjective element into the situation. But, as Heyting rightly observes, this subjective element is eliminated if we agree to concentrate upon such properties of choice sequences as appear after a finite number of choices. This attitude implies, however, a radical change in the semantic notions. The classical rules determine [...] the validity or non-validity of a formula *on each branch* separately [...] These difficulties vanish, if we agree to determine validity or non-validity, not on individual branches, but collectively on all those branches which have a certain initial element in common, that is, *on a subtree.*

In this passage, he explains his use of "submodel" (bar).

Beth had to avoid some classical principles as compactness (ms. E.W. Beth, "Semantische Tafeln für die intuitionistische Prädikatenlogik erster Ordnung", lecture IV. Österr. Mathematikerkongress, Wien, September 17–22, 1956):

> If the construction of tableaux is interpreted as an attempt to construct a counter-model, then just as before we have to use an intuitionistic non-acceptable compactness argument for the proof that for every non-closed semantic tableau a corresponding counter-model actually exists.

The original semantic tableaux had been *countermodel-constructions*. For intuitionistic purposes these were now transformed into an operation of 'fitting in', as witnessed by a letter from Beth to Heyting, dated November 18, 1955 (this letter must have been written after his Paris lectures, but nevertheless he did not modify Beth (1958) in this respect):

> Let us think of a certain (given) model of the A's, and also of a model, as yet undetermined, which is a union of submodels as intended above. Call these models $\mathcal{M}$ and $\mathcal{N}$. Now we are going to split $\mathcal{M}$ and $\mathcal{N}$ ever farther into submodels. Splitting of $\mathcal{M}$ yields a conjunctive splitting of the tableau, splitting of $\mathcal{N}$ a disjunctive splitting. The construction succeeds, if we have split $\mathcal{M}$ into parts, each of which fits into some piece of $\mathcal{N}$.

The two "truth values" here are "true" and "not yet decided". The combination $< \mathcal{M}, \mathcal{N} >$ is a fan. Beth used Brouwer's fan-theorem with its restrictions as a framework for his intuitionistic completeness proof.

It is possible to have infinite branches in a submodel; these are produced by Beth's cyclic permutation of formulas. Beth introduced trunks (truncated trees) with a finite depth, in order to compare the conjuctive and the disjunctive part of a model up to a certain depth.

Beth formulated it in his lecture "Intuitionistic predicate logic" at Cornell University in 1957 as follows:

> This [effective] procedure [from $\mathcal{M}$ to $\mathcal{N}$ as above] cannot applied to complete trees, it must work on trunks of sufficient length. As trunks of models cannot (always) be distinguished from trunks of semi-models, the procedure must work as well on trunks of semi-models which are sufficiently like models. [...] the procedure must lead to one of the following results: (1) the trunk cannot be extended into a model of K [i.e., the conjunctive part]; (2) every extension of the trunk into a model of K is also a model of L [i.e., the

> disjunctive part]. hence, duplicating Brouwer's second proof of the so-called fundamental theorem of finitary spreads, we can establish the existence of an upper bound k' for all "sufficient lenghts".

5 The paper from 1956 and afterwards

Beth published in 1956 a paper (some parts of which we used already) in the Proceedings of the Royal Dutch Academy of Sciences in which he gives a correct definition of validity for his models, and presents a completeness proof in two versions, a classical and an intuitionistic one. Beth's arguments are not very easy to follow; in part this is due to a somewhat heavy notation. His classical proof was accepted as in essence correct. But several critics complained about the purported intuitionistical argument. They either considered it as unintelligible (Heyting) or flawed. Improving Beth's proposals took a lot of time, but as Kreisel said to Beth in his letter of December 19, 1957 (Reading):

> I regard your work as a very important contribution to the subject. For this reason it seems to me desirable to give a flawless proof of your result [i.e. 'duplicating Brouwer's proof'].

After nearly three years Kreisel told Beth in a letter from Paris, dated November 11, 1960, about his work:

> During the summer we prepared at Stanford a *Project report* concerning the completeness of intuitionistic logic including (a) [...], (c) a very detailed version of your completeness proof with a full proof of its applicability to decidable subsystems, (d) [...]

Under (c) one finds the reference to Dyson & Kreisel (1961). Beth's reaction was a short, laconic one (letter to Kreisel, December 5, 1960):

> I shall be very much interested to see your *Project report*. If I do not find time to read it carefully, it would certainly provide excellent material for a seminar either by myself or by Heyting and me.

We have not been able to find any later remark by Beth on this report.

Beth never took the time to thrash out the criticisms in detail. In the beginning he claims that the criticism of his proof should really be taken as criticism of Brouwer's argument for the fan theorem. An example you can find in Beth's letter to G. Kreisel of December 23, 1957:

> With respect to the intuitionistic proof, my own attitude is ambivalent. This proof seems to belong to intuitionistic higher-order logic, for which no formal standards are available at present. On the other hand, all proofs of the fan theorem which I have seen seem to be fallacious. Looking at the matter under this aspect, 'duplicating Brouwer's proof' [a remark made by Kreisel in his letter to Beth, (Reading), December 18, 1957 (p. 3)] is a polite expression for imitating Brouwer's fallacy.

Kleene and Kreisel had discussed Beth's results. Heyting wrote in a letter to Beth of January 19, 1958 (Notre Dame) as a significant remark:

> It appeared that your paper on intuitionistic logic had led to a rather thorough dicussion between Kleene and Kreisel. New to me in this was, that your interpretation of negation and implication only holds if one permits

absolutely free choice sequences, and not the sequences determined by a law.

If the flaw in Beth's alleged intuitionistic completeness proof did not reside in the fan theorem per se, what was then the real weakness? Beth's earlier version (e.g., Beth (1959*a*), p. 459) of his intuitionistic completeness was: "If a sequent $C \Rightarrow D$ holds true intuitionistically, then it is derivable in the formal system $F0$ [i.e. Beth's syntactical part]." At the end of the intuitionistic section in his 'Foundations', p. 461, he changed this into: "If all models $\mathcal{M}$ which fulfil the conjunctive C also fulfil the disjunctive D [of the sequent $C \Rightarrow D$], then it [i.e. the sequent] is derivable in the formal system $F0$." The discussion thereafter concerns the status of "all models". Beth wrote December 8, 1957 to Kreisel (at that time Beth thought his 'Foundations' was ready for printing):

> The hypothesis in the completeness theorem contains the quantifier "all models", whereas (in general) the species of all models of a given conjunctive cannot be represented by a spread. My paper contains a method for embedding all models of a given conjunctive in a spread which in addition contains semi-models.
>
> But presumably the species of all models can also be embedded in other spreads and, as I have shown in my paper in Amsterdam [i.e. Beth (1959*b*)] the meaning of the quantifier "all models" may depend on the choice of the spread in which these models are embedded.[...] so the matter can only be settled on the basis of a discussion on intuitionistic higher-order logic, which seems to be an immensely intricate subject.

Not long after the first interchanges with G. Kreisel (and S.C. Kleene?) Beth had come to realize that the crux of the matter was his application of the fan theorem to a spread of initial parts of potential models; not all of these initial parts belonged to proper models, some of them could only lead to inconsistent models. In other words, "validity in all Beth-models" had to be read as "validity in all potential models" in order to make sense of Beth's argument.

Beth never found time to integrate critical remarks (Beth (1959*a*), p. 461: "[...] objections which have been raised from different viewpoints, by A. Heyting and by K. Gödel and G. Kreisel") in his work. Certainly he did have no time just before the printing of his 'Foundations'; so he wrote to Kreisel, December 23, 1957 in relation to his "Foundations of mathematics":

> So it seems better to include a summary of my material in its present state and to mention the fact that certain objections have been raised and my own attitude on the above lines. I plan to go into the problem of intuitionistic higher-order logic when the book is ready.

Beth never carried out these plans; so it was left to Veldman (1976) and de Swart (1976), much later, to rediscover this idea and make use of it.

References

Beth, E.W. (1947), 'Semantic considerations on intuitionistic mathematics', *Indagationes Mathematicae* **9**, 572–577.

Beth, E.W. (1955*a*), 'Remarks on natural deduction', *Indagationes Mathematicae* **17**, 322–325. (dedicated to Robert Feys).

Beth, E.W. (1955*b*), 'Semantic entailment and formal derivability', *Med. Konkl. Nederl. Akad. v. Wetensch., Nieuwe Reeks* **18**(13), 309–342.

Beth, E.W. (1956*a*), *l'Existence en mathématiques*, number 10 *in* 'Collection de logique mathématiques, série A', Gauthier-Villars/Nauwelaerts, Paris/Louvain. (66 p.). (Beth's lectures at Sorbonne University (Paris), March 29 – April 2, 1954).

Beth, E.W. (1956*b*), 'Semantic construction of intuitionistic logic', *Koninklijke Nederlandse Akademie van Wetenschappen, Mededelingen, Nieuwe Reeks* **19**(11), 357–388.

Beth, E.W. (1957), Intuitionistic predicate logic, *in* 'Summaries of talks presented at the Summer Institute for Symbolic Logic in 1957', Vol. 1, Cornell University, pp. 113–121. (1960^2).

Beth, E.W. (1958), Construction sémantique de la logique intuitioniste, *in* 'Le raisonnement en mathématique et en sciences experimentales (Paris, 26 Sept.–1 Oct. 1955)', number 70 *in* 'Colloques Intern. du C.N.R.S.', pp. 77–83. (A. Heyting, 'Interventions', related to Beth's lecture: p. 84).

Beth, E.W. (1959*a*), *The foundations of mathematics. A study in the philosophy of sciences*, Studies in Logic, North Holland, Amsterdam. (XXVI + 722 pp.). (1965^2 revised, North Holland, (XXVIII + 741 pp.). (1966^3, Harper).

Beth, E.W. (1959*b*), Remarks on intuitionistic logic, *in* A. Heyting, ed., 'Constructivity in mathematics', Studies in Logic, (Colloquium constructivity in mathematics, Amsterdam, August 1957), North Holland, Amsterdam, pp. 15–25.

Brouwer, L.E.J. (1926), 'Die intuitionistische Form des Heine-Borelschen Theorems', *Koninkl.Akad.Wetensch., Proceedings* **29**(6), 866–867. (=De intuïtionistische vorm van het theorema van Heine-Borel, K.A.W., Verslagen 35, pp. 667–678).

de Jongh, D.H.J. & van Ulsen, P. (1998), Beth's nonclassical valuations, Technical Report LP-1998-12, Institute for logic, language and computation, Universiteit van Amsterdam.

de Swart, H.C.M. (1976), 'Another intuitionistic completeness proof', *The Journal of Symbolic Logic* **41**, 644–662. (received September 24, 1974).

Dyson, V.H. & Kreisel, G. (1961), Analysis of Beth's semantic construction of intuitionistic logic, Technical Report 3, Applied mathematics and statististical laboratories, Stanford University, Stanford. (Office of Ordnance Research, Contract No. DA-04-200-ORD-997).

Heyting, A. (1931), 'Die intuitionistische Grundlegung der Mathematik', *Erkenntnis* **2**, 106–115. (=Annalen der Philosophie 10).

Heyting, A. (1934), *Mathematische Grundlagenforschung. Intuitionismus, Beweistheorie*, Ergebnisse der Mathematik und ihre Grenzgebiete, Springer, Berlin.

Heyting, A. (1956), *Intuitionism, An introduction*, Studies in logic, North Holland. (1966^2, 1971^3, both revised).

Jaskowski, S. (1936), Recherches sur le système de logique intuitioniste, *in* 'Actes du Congrès International de Philosophie Scientifique (Sorbonne, Paris, 1935), VI, Philosophie des Mathématiques', Hermann, Paris, pp. 58–61.

Kleene, S.C. (1945), 'On the interpretation of intuitionistic number theory', *The Journal of Symbolic Logic* **10**, 109–124.

Kleene, S.C. (1948), 'Review of E.W. Beth, 'Semantical considerations on intuitionistic mathematics', Indag. Math 9, 572–577', *Journal of Symbolic Logic* **13**, 173.

Kleene, S.C. (1952), *Introduction to metamathematics*, number 1 *in* 'Bibliotheca Mathematica', North Holland, Amsterdam. (1959^2).

Kolmogorov, A.N. (1932), 'Zur Deutung der intuitionistischen Logik', *Mathematische Zeitschrift* **35**, 58–65.

Odifreddi, P. (1989), *Classical recursion theory, The theory of functions and sets of natural numbers*, number 125 *in* 'Studies in Logic', North Holland, Amsterdam. (1992^2).

Stone, M.H. (1937), 'Topological representations of distributive lattices and Brouwerian logics', *Časopis Pro Pěstvování Matematiky a Fysiki* **67**, 1–25.

Tarski, A. (1935), 'Der Wahrheitsbegriff in den formalisierten Sprachen', *Studia Philosophica* **1**, 261–405.

Tarski, A. (1938), 'Der Aussagenkalkül und die Topologie', *Fundamenta Mathematicae* **31**, 103–134.

Troelstra, A.S. & van Dalen, D. (1988), *Constructivism in Mathematics, II*, number 123 *in* 'Studies in logic', NorthHolland, Amsterdam.

Veldman, W. (1976), 'An intuitionistic completeness theorem for intuitionistic predicate logic', *The Journal of Symbolic Logic* **41**, 159–166. (received May 20, 1974).

Beth semantic tableaux for intuitionistic logic is developed in the chapter. The chapter describes consistency property: in a standard formalization of Heyting's predicate calculus, the axioms are all valid, and the rules preserve validity. View.Â In this paper we propose a Kripke-style semantics for second order intuitionistic propositional logic and we provide a semantical proof of the disjunction and the explicit definability property. Moreover, we provide a tableau calculus which is sound and complete with respect to such a semantics. (Â© 2004 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim). Intuitionistic logic, sometimes more generally called constructive logic, is a system of symbolic logic that differs from classical logic by replacing the traditional concept of truth with the concept of constructive provability.Â Although intuitionistic logic retains the trivial propositions from classical logic, each proof of a propositional formula is considered a valid propositional value, thus by Heyting's notion of propositions-as-sets, propositional formulae are (potentially non-finite) sets of their proofs.Â The discovery of E.W. Beth's semantics for intuitionistic logic by A.S. Troelstra and P. van Ulsen. Expressing Database Queries with Intuitionistic Logic (FTP one-click download) by Anthony J. Bonner. L. Thorne McCarty. Hilbertâ€™s choice operators Ï„ and Îµ, when added to intuitionistic logic, strengthen it. In the presence of certain extensionality axioms they produce classical logic, while in the presence of weaker decidability conditions for terms they produce various superintuitionistic intermediate logics. In this thesis, I argue that there are important philosophical lessons to be learned from these results. To make the case, I begin with a historical discussion situating the development of Hilbertâ€™s operators in relation to his evolving program in the (...) foundations of mathematics and in relation to ph