# Topological Methods in Combinatorics

Samuel Kolins

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# Chapter 1

# Introduction

The course will be divided into two main sections. For the first half of the course we will look at applications of topological results to prove combinatorial theorems as well as some uses of combinatorics to prove topological results. The main reference for this part of the course will be J. Matoušek's book *Using the Borsuk-Ulam Theorem, Lectures on Topological Methods in Combinatorics and Geometry* [4]. The second part of the course will look at discrete Morse Theory. A good reference for this section will be Dmitry Kozlov's text *Combinatorial Algebraic Topology* [2].

## 1.1 Overview of Part One of the Course

### 1.1.1 The Borsuk-Ulam Theorem

In order to state the Borsuk-Ulam Theorem we need the idea of an antipodal map, or more generally a $\mathbb{Z}_2$ map. Note that in this class, all maps between topological spaces are continuous unless otherwise specified.

**Definition 1.1.** *A $\mathbb{Z}_2$ space $(X, \epsilon)$ is a topological space $X$ with a $\mathbb{Z}_2$ action.*

*Let $(X, \epsilon)$ and $(Y, \tau)$ be $\mathbb{Z}_2$ spaces. Then $f : (X, \epsilon) \to (Y, \tau)$ is a $\mathbb{Z}_2$-map if $f \circ \epsilon = \tau \circ f$.*

The classical example of a $\mathbb{Z}_2$ space is the $d$-sphere $S^d$ with the $\mathbb{Z}_2$ action $\epsilon(x) = -x$. An *antipodal map* is a $\mathbb{Z}_2$ map between two spheres each with this standard $\mathbb{Z}_2$ action. The Borsuk-Ulam Theorem states that a particular type of antipodal map does not exist.

**Theorem 1.2** (Borsuk-Ulam)**.** *There does not exist an antipodal map from $S^{d+1}$ to $S^d$.*

It is not hard to convince oneself that the theorem holds for the case $d = 0$, where you have a (continuous) map from the circle to two isolated points. We will present a few different proofs of the theorem in the following chapter.

We now discuss some of the ways that we will use the Borsuk-Ulam Theorem in the course.

### 1.1.2 Applications in Topology (with Further Applications in Geometry)

We will use the Borsuk-Ulam theorem to derive other topological results. These include the "Ham Sandwich Theorem" as well as Brouwer's fixed point theorem. Let $B^n$ be the $n$-dimensional unit ball.

**Theorem 1.3** (Brouwer's Fixed Point Theorem)**.** *If $f : B^n \to B^n$ then there exists $x \in B^n$ such that $f(x) = x$.*

We can apply Brouwer's fixed point theorem to study the combinatorics of the game hex. In hex, two players (blue and red) take turns coloring one hexagon of a grid of hexagons. After all the hexagons are filled, blue wins if there is a path of blue hexagons connecting the top and bottom of the grid and red wins if there is a path of red hexagons connecting the left and right sides of the board. The left side of Figure 1.1.2 shows a completed game that was won by red.

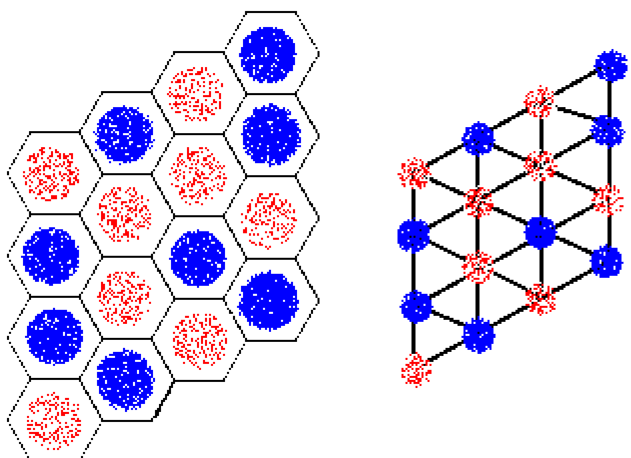

Figure 1.1: Hex board and dual hex board

An alternative but equivalent way to look at the hex game is to look at the dual to the hexagonal grid. Each hexagon in the original grid corresponds to vertex and two vertices are connected by an edge if the corresponding hexagons were adjacent. Now the players color the vertices with blue winning if there is a path from the top to the bottom of the grid through only blue vertices and red winning if there is a path from the left side to the right side of the grid through only red vertices. The right hand side of Figure 1.1.2 shows the dual version of the same hex game.

A basic question about hex is if there must always be a winning player. From the Jordan curve theorem, if any player wins then the winning path separates the playing area, so the second player can not also win. We will use Brouwer's fixed point theorem to show that one player must always win.

**Theorem 1.4** (Hex Theorem)**.** *In hex, there exists a crossing path for exactly one of the players.*

It remains to prove that there is a winning path for at least one of the players; the following argument proves this part of the theorem.

*Proof.* We work in the setting of the dual game. Assume that no winning path exists. Partition the vertices into four sets:

$$\begin{aligned}
S_0 &= \text{ blue vertices reached from the lower edge by a blue path}\\
N_0 &= \text{ blue vertices not in } S_0\\
W_0 &= \text{ red vertices reached from the left hand side by a red path}\\
E_0 &= \text{ red vertices not in } W_0
\end{aligned}$$

Let $e_1$ be a rightward shift by one vertex (parallel to the top and bottom of the board) and $e_2$ be an upward shift by one vertex. Then define the function $f$ on the vertices of the grid by

$$f(v) = \begin{cases} v + e_2 & v \in S_0 \\ v - e_2 & v \in N_0 \\ v + e_1 & v \in W_0 \\ v - e_1 & v \in E_0 \end{cases}$$

Since we are assuming that no winning path exists, the map $f$ is well defined.

Now consider any triangle in our grid with vertices $v_1, v_2, v_3$. Each point $x$ in our triangle can be written uniquely as a convex combination $x = \sum x_i v_i$ where $x_i \geq 0$ and $\sum x_i = 1$. We can extend the map $f$ linearly on each triangle by letting $f(x) = \sum x_i f(v_i)$. This gives a map $f$ from the entire filled grid to itself.

Since the filled grid is topologically a 2-ball, we can apply Brouwer's Fixed Point Theorem. So there is some point $x$ in our grid such that $f(x) = x$. Writing $x$ as a convex combination of vertices as above and letting $\epsilon_i \in \{\pm e_1, \pm e_2\}$ we have

$$\begin{aligned}
\sum x_i v_i &= \sum x_i (v_i + \epsilon_i) \\
0 &= \sum x_i \epsilon_i \qquad (1.1)
\end{aligned}$$

We know that there is some $x_j > 0$. Without loss of generality, let $\epsilon_j = e_1$. Then in order to get the right-hand side of (1.1) to be zero we need some other $\epsilon_i$ to be equal to $-e_1$. So we have two vertices in the same triangle one of which is in $E_0$ and the other in $W_0$. However, all vertices in a triangle are connected by edges, so we have contradicted the definition of our partition. Therefore we must have a winning path. □

### 1.1.3 Lower Bounds on Chromatic Numbers of Graphs

**Definition 1.5.** *Let $G = (V, E)$ be a graph. Let $[k] = \{1, 2, \ldots, k\}$. Define the chromatic number $\chi(G)$ by*

$$\chi(G) = \min\{k : \exists\, c : V \to [k] \text{ such that if } \{x, y\} \in E \text{ then } c(x) \neq c(y)\}$$

*Such a map $c$ is called a $k$-coloring of $G$.*

Our goal will be to determine the chromatic number of various graphs. One graph in which we will be interested is the Kneser graph. We use the notation $\binom{[n]}{k}$ for the collection of all size $k$ subsets of $[n]$.

**Definition 1.6.** *Let $n \geq 2k$. Then the Kneser graph $K_{n,k} := (V, E)$ where $V \in \binom{[n]}{k}$ and $\{A, B\} \in E$ if and only if $A \cap B = \emptyset$.*

The special case of $K_{5,2}$ is the well known Petersen Graph. Figure 1.1.3 shows this graph with the vertices labeled by the sets to which they correspond.

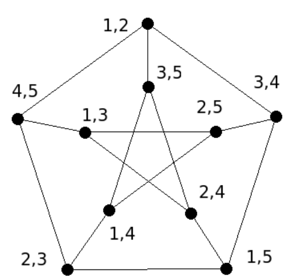


Figure 1.2: The Petersen graph, $K_{5,2}$

One easy way to color a Kneser graph is to assign a color to each element of $[n]$ and then color each set $A \in V$ by the color of the minimal element of $A$. In order to use even fewer colors, note that any two $k$ subsets of $\{n - 2k + 2, n - 2k + 3, \ldots, n\}$ must intersect and hence can not be connected by an edge in the Kneser graph $K_{n,k}$. Therefore, we can color all of these sets a single color, and color the remaining sets by their minimal elements as above. This gives a coloring using only $n - 2k + 2$ colors.

We will prove that in fact this coloring in minimal, i.e. $\chi(K_{n,k}) = n - 2k + 2$. This result was conjectured by Kneser in 1955 [1] and proved by Lovász in 1978 [3].

### 1.1.4 Topological Extension of Classical Theorems in Convexity

Another goal of the course will be to prove topological versions of some famous theorems in convexity theory. Here we mention two of these theorems.

**Theorem 1.7** (Radon). *Let $X \subseteq \mathbb{R}^d$, $|X| = d + 2$. Then there exists a partition $X = A \sqcup B$ such that $\mathrm{conv}(A) \cap \mathrm{conv}(B) \neq \emptyset$.*

Let $\sigma^d$ be the $d$-dimensional simplex. Then we can generalize Radon's theorem to the following topological statement

**Theorem 1.8** (Topological Radon). *Let $f : \sigma^{d+1} \to \mathbb{R}^d$. Then there exist disjoint faces $A, B$ of $\sigma^{d+1}$ such that $f(A) \cap f(B) \neq \emptyset$.*

By taking $f$ to be a linear map we can recover the original Radon's Theorem from the topological version.

Radon's theorem is a special case (r=2) of Tverberg's Theorem.

**Theorem 1.9** (Tverberg). *Let $d \geq 1$ and $r \geq 2$. If $X \subseteq \mathbb{R}^d$ with $|X| = (d+1)(r-1)+1$ then there exists a partion $X = \sqcup_{i=1}^r A_i$ such that $\cap_{i=1}^r \mathrm{conv}(A_i) \neq \emptyset$.*

The following topological version is conjectured to be true.

**Conjecture 1.10** (Topological Tverberg). *Let $f : \sigma^{(d+1)(r-1)} \to \mathbb{R}^d$. Then there exist pairwise disjoint faces $\{A_i\}_{i=1}^r$ of $\sigma$ such that $\cap_{i=1}^r A_i \neq \emptyset$.*

This conjecture in know to hold only in the cases where $r$ is a prime power. This will follow from an extension of the Borsuk-Ulam theorem to prime power actions.

### 1.1.5 Applications to Non-Embeddability

You probably know that the graphs $K_5$ and $K_{3,3}$ are not planar, i.e. can not be embedded in $\mathbb{R}^2$. We will be interested in non-embeddability results in higher dimesions. The complexes in which we will be interested will be the Van-Kampen Flores complexes, $\sigma_{\leq d}^{2d+2}$. We define $\sigma_{\leq d}^{2d+2}$ to be the $d$-skeleton of the $(2d+2)$-simplex, i.e. all faces of $\sigma^{2d+2}$ of dimension less than or equal to $d$. The case $\sigma_{\leq 1}^4$ is just the complete graph $K_5$, which we know does not embed in $\mathbb{R}^2$. We will show that in general $\sigma_{\leq d}^{2d+2}$ does not embed in $\mathbb{R}^{2d}$.

In order to prove this result we will need a space with a free $\mathbb{Z}_2$ action. Letting $X = \sigma^{2d+2}_{\leq d}$, the space $X \times X$ has a natural $\mathbb{Z}_2$ action given by $(x, y) \to (y, x)$. However, in order to make this action free we remove the fixed points and consider the space $X \times X - \{(x, x) : x \in X\}$.

## 1.2 A Combinatorial Proof of the Brouwer's Fixed Point Theorem

Our next goal is a combinatorial proof of Brouwer's fixed point theorem. The first step in this proof is Sperner's Lemma.

**Lemma 1.11** (Sperner's Lemma)**.** *Let $\sigma^n = \text{conv}\{v_0, v_1, \ldots, v_n\} \subseteq \mathbb{R}^n$ be an $n$-dimensional simplex. Let $T$ be a triangulation of $\sigma^n$. Let $\phi : V(T) \to \{0, 1, \ldots, n\}$ be an $(n+1)$-coloring of the one-skeleton of $T$ such that for any vertex $v \in V(T)$ if $v \in \text{conv}\{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\}$ then $\phi(v) \in \{i_0, i_1, \ldots, i_k\}$. Then there exists an odd number of $n$-simplexes $\tau \in T$ such that $\phi(V(\tau)) = \{0, 1, \ldots, n\}$.*

We call a $k$-simplex $\tau$ colorful if $\phi(V(\tau)) = \{0, 1, \ldots, k\}$.

*Proof.* Our proof is by induction on $n$. The base case $n = 0$ is trivially true. Alternatively, the case $n = 1$ is the case of a partitioned line segment with endpoints of opposite colors. There must be an odd number of color changes when moving along the segment from one endpoint to the other, giving an odd number of the desired type of simplexes.

We now prove the inductive step. Denote by $a_k$ the number of colorful $k$-simplexes in $T$. We want to show that $a_n$ is odd.

Let $A$ be the set of all pairs $(\sigma, \tau)$ such that $\tau$ is an $n$-simplex, $\sigma$ is an $(n-1)$-simplex, $\sigma \subseteq \tau$, and $\sigma$ is colorful. We count $|A| \pmod 2$ in two different ways, once from the perspective of the $\sigma$ and once from the perspective of the $\tau$.

We begin by considering possible cases for the face $\sigma$. If $\sigma$ is an interior face then there are exactly two faces $\tau$ that can be paired with $\sigma$ to create an element of $A$. Therefore, interior faces do not contribute to $|A| \pmod 2$. From our condition on $\phi$, $\sigma$ can not be a subset of any boundary face that contains the vertex $\{n\}$. If $\sigma$ is on the boundary face $\{0, 1, \ldots, n-1\}$ then there is exactly one $\tau$ that can be paired with $\sigma$. So the size of $|A|$ is equal (mod 2) to the number of colorful $(n-1)$-simplexes in the face $\{0, 1, \ldots, n-1\}$. However, the face $F = \{0, 1, \ldots, n-1\}$ and the coloring $\phi|_F$ themselves satisfy the conditions of Sperner's lemma, so by the inductive hypothesis we know that there are an odd number of colorful $(n-1)$-simplexes in $\{0, 1, \ldots, n-1\}$. Hence $|A|$ is odd.

Now we consider the possible cases for $\tau$. If $\phi(V(\tau)) \not\supseteq \{0, 1, \ldots, n-1\}$ than no $(n-1)$ simplex contained in $\tau$ can be colorful, so $\tau$ does not contribute to $|A|$. If $\tau$ is colorful then there is exactly one colorful $(n-1)$-simplex contained in $\tau$, namely the simplex whose vertices are colored with $\{0, 1, \ldots, n-$

1}. These $\tau$ contribute a total of $a_n$ pairs to $|A|$. If $\phi(V(\tau)) \supseteq \{0, 1, \ldots, n-1\}$ but $\tau$ is not colorful then there is a single pair of vertices in $\tau$ with the same color and hence there are exactly two colorful $(n-1)$-simplexes contained in $\tau$. Therefore we have $|A| = a_n \pmod 2$. Combining this with the above work proves that $a_n$ is odd, as desired. □

We now use Sperner's lemma to prove Brouwer's fixed point theorem.

*Proof of Brouwer's fixed point theorem (Theorem 1.3).* Since the ball $B^n$ is homeomorphic to the simplex $\sigma^n$ we take $f : \sigma^n \to \sigma^n$. Let $\{v_0, v_1, \ldots, v_n\}$ be the vertices of $\sigma^n$. Let $T$ be a triangulation of $\sigma^n$ and let $x \in \sigma^n$. We can write $x$ uniquely as a convex combination of the $v_i$, $x = \sum_{i=0}^{n} \lambda_i(x) v_i$. Define

$$S_i := \{x \in \sigma^n : \lambda_i(f(x)) \leq \lambda_i(x)\}.$$

Note that since $\sum \lambda_i(x) = \sum \lambda_i(f(x)) = 1$, the $S_i$ cover $\sigma^n$. Similarly, along any face $\tau$ of the boundary of $\sigma^n$, the $\{S_i : v_i \in \tau\}$ cover $\tau$. Therefore, we can define a coloring $\phi$ of the vertices of $T$ by taking $\phi(v)$ to be equal to some $i$ such that $v \in S_i$ and further we can define $\phi$ along the boundary faces in such a way so that the resulting triangulation satisfies the hypothesis of Sperner's lemma.

Using this coloring, we must have some colorful $n$-simplex $\tau \in T$. Label the vertices of $\tau$ by $\{u_0, u_1, \ldots, u_n\}$ where $\phi(u_i) = i$. (Note that the $u_i$ are points in $\sigma^n$ and not components). We have $\lambda_i(f(u_i)) \leq \lambda_i(u_i)$.

Now instead of considering a single triangulation we consider a sequence $\{T_k\}_{k=1}^{\infty}$ of triangulations. Define the mesh of a triangulation by

$$\text{mesh}(T_k) := \max\{\text{diam}(F) : F \text{ is a face in } T_k\}.$$

We choose our sequence $\{T_k\}$ such that $\lim_{k\to\infty} \text{mesh}(T_k) = 0$.

Repeating the above process for each $T_k$ let $\tau_k$ be the resulting simplex and let the $u_i^k$ be the vertices of $\tau_k$. The vectors $(u_1^k, u_2^k, \ldots, u_n^k)$ are all elements of the compact space $\sigma^n \times \sigma^n \times \cdots \times \sigma^n$. Therefore, some subsequence must converge to a point $(w_1, w_2, \ldots, w_n)$ with $\lambda_i(f(w_i)) \leq \lambda_i(w_i)$ for all $i$. Furthur, from our assumption about $\text{mesh}(T_k)$ we know that all the $w_i$ are in fact the same point; call this point $w$. We have $\lambda_i(f(w)) \leq \lambda_i(w)$ for all $i$ and $\sum \lambda_i(f(w)) = \sum \lambda_i(w)$, hence $\lambda_i(f(w)) = \lambda_i(w)$ for all $i$ and $w$ is a fixed point of $f$, as desired. □

It is worth considering how to conclude Sperner's lemma from Brower's fixed point theorem. Additionally, how can you obtain Brouwer's fixed point theorem for disks from the hex theorem?

## 1.3 Realizations of Simplicial Complexes

Let $K$ be a simplicial complex and let $\|K\|$ be the geometric realization of $K$. Recall that the dimension of $K$ is $\dim K := \max_{F\in K}(|F|-1) = \max_{F\in K} \dim \operatorname{aff}(\|F\|)$. If $K$ has $n$ vertices then $K$ can easily be realized in $\mathbb{R}^n$ by taking the vertices of $\|K\|$ to the the unit basis vectors. The following theorem gives a more efficient realization.

**Theorem 1.12.** *Let $K$ be a (finite) abstract simplicial complex of dimension $d$. Then $\|K\| \hookrightarrow \mathbb{R}^{2d+1}$.*

*Proof.* First note that if $f : V(K) \to \mathbb{R}^m$ is an injective map such that for all faces $F, G \in K$ the image $f(F \cup G)$ is an affinely independent set then the identification

$$F \mapsto \sigma_F := \operatorname{conv}_{v\in F} f(v)$$

gives an embedding $\|K\| \hookrightarrow \mathbb{R}^m$. So if we can find an infinite collection of points in $\mathbb{R}^{2d+1}$ such that any $2d+2$ points are affinely independent then we will have proved the desired result.

Define the moment curve $\mathcal{M}_m := \{(t, t^2, t^3, \ldots, t^m) : t \in \mathbb{R}\} \subseteq \mathbb{R}^m$. We will show that any $m+1$ points on $\mathcal{M}_m$ are affinely independent, completing the proof. We do this by showing that for any hyperplane $H \subseteq \mathbb{R}^m$, $|H \cap \mathcal{M}_m| \leq m$. We can express our hyperplane as

$$H = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} a_i x_i = b\}$$

for some fixed $a_1, \ldots, a_m, b \in \mathbb{R}$ where not all the $a_i$ are zero. Then $H \cap \mathcal{M}_m = \{(t, t^2, \ldots, t^m) : \sum_{i=1}^{m} a_i t^i = b, t \in \mathbb{R}\}$. This is a non-trivial polynomial and hence has at most $m$ roots. □

## 1.4 Posets and Order Complexes

**Definition 1.13.** *Let $(P, <)$ be a poset. Then the order complex $\Delta(P)$ is the simplicial complex whose faces are $\{x_1, x_2, \ldots, x_k\}$ where $x_1 < x_2 < \cdots < x_k$ in $P$.*

Let $K$ be a simplicial complex. Then $K$ has a natural poset structure with the faces ordered by inclusion. We write this as $P(K) = (K - \emptyset, \subseteq)$. What happens when you take $\Delta(P(K))$?

**Definition 1.14.** *For a simplicial complex $K$, $\Delta(P(K))$ is called the Barycentric subdivision of $K$ and is denoted* $\operatorname{sd}(K)$.

**Exercise 1.4.1.** Prove that $\|K\| \cong \|\Delta(P(K))\|$.

We now briefly consider maps between posets and maps between simplicial complexes.

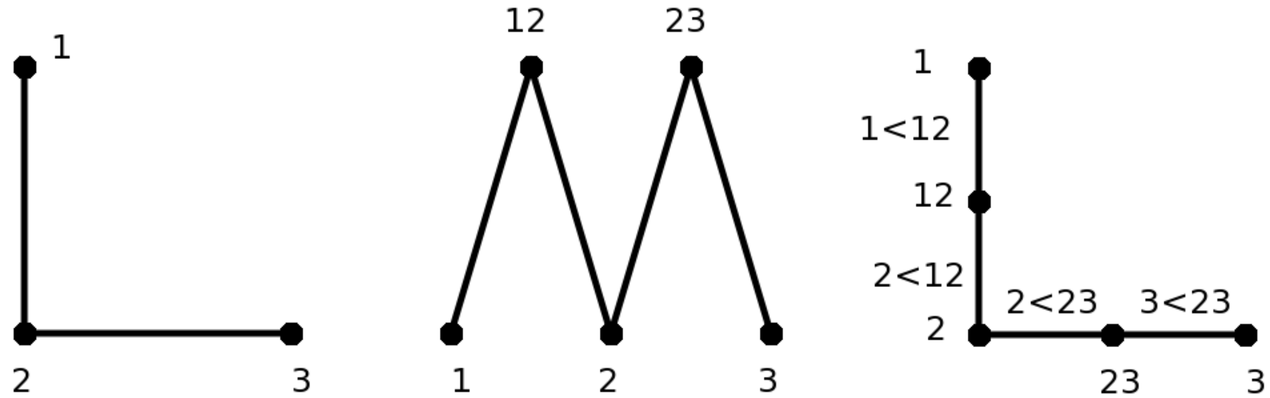


Figure 1.3: A complex $K$, poset $P(K)$, and Barycentric subdivision $\Delta(P(K)) = sd(K)$.

**Definition 1.15.** *Let $(P_1, <_1)$ and $(P_2, <_2)$ be posets. A map $f : P_1 \to P_2$ is order preserving (also called a poset map) if $x <_1 y$ implies $f(x) <_2 f(y)$.*

**Definition 1.16.** *Let $K, L$ be simplicial complexes. A map $F : V(K) \to V(L)$ is a simplicial map if for all $\sigma \in K$, $f(\sigma) \in L$.*

The next proposition follows immediately from these definitions.

**Proposition 1.17.** *Let $P_1$ and $P_2$ be posets. If $f : P_1 \to P_2$ is order preserving then the induced map $f : V(\Delta(P_1)) \to V(\Delta(P_2))$ is a simplicial map.*

## 1.5 Brouwer's Fixed Point Theorem from Borsuk-Ulam

We conclude this chapter by giving a proof of Brouwer's Fixed Point Theorem (Theorem 1.3) using the Borsuk-Ulam Theorem. In the next chapter we will prove many equivalent versions of the Borsuk-Ulam Theorem. Here we use the statement that there does not exist a map $F : B^n \to \partial B^n$ that is antipodal on $\partial B^n$.

*Proof of Brouwer's Fixed Point Theorem using Borsuk-Ulam.* Let $f : B^n \to B^n$ with no fixed point. Define $g : B^n \to \partial B^n$ by letting $g(x)$ be the point where the open ray $\overrightarrow{f(x)\,x}$ intersects $\partial B^n$. The map $g$ is well defined because $f$ has no fixed points. In is also easy to see that $g$ is continuous. On $\partial B^n$ the map $g$ is the identity map and hence antipodal. □

# Chapter 2

# The Borsuk-Ulam Theorem

## 2.1 Equivalent Statements of the Borsuk-Ulam Theorem

We begin our detailed study of the Borsuk-Ulam Theorem by presenting six equivalent versions and proving that they are in fact all equivalent. In the following section we will show that these equivalent statements are in fact true.

**Theorem 2.1** (Borsuk-Ulam). *For all $n \geq 0$ the following are equivalent and true*

*(1a) For all $f : S^n \to \mathbb{R}^n$ there exists $x \in S^n$ such that $f(x) = f(-x)$.*

*(1b) For all antipodal maps $f : S^n \to \mathbb{R}^n$ there exists $x \in S^n$ such that $f(x) = 0$.*

*(2a) There is no antipodal map $f : S^n \to S^{n-1}$.*

*(2b) There is no map $f : B^n \to S^{n-1}$ such that $f|_{\partial B^n}$ is antipodal.*

*(C) For every closed cover $F_1, \dots, F_{n+1}$ of $S^n$ there exists an $i$ such that $F_i \cap (-F_i)$ is non-empty (i.e. there exists a pair of antipodal points in $F_i$).*

*(O) For every open cover $U_1, \dots, U_{n+1}$ of $S^n$ there exists an $i$ such that $U_i \cap (-U_i)$ is non-empty.*

We will show the equivalences $(1a) \iff (1b) \iff (2a) \iff (2b)$, $(1a) \Rightarrow (C) \Rightarrow (2a)$, and $(C) \iff (O)$.

*Equivalence of Conditions in Theorem 2.1.*

(1a) $\Rightarrow$(1b): Let $f : S^n \to \mathbb{R}^n$ be an antipodal map, so $f(-a) = -f(a)$ for all $a \in S^n$. Then by (1a) there exists $x \in S^n$ such that $f(-x) = f(x)$. Therefore we have $-f(x) = f(-x) = f(x)$, so $f(x) = 0$, as desired.

(1b) $\Rightarrow$(1a): Let $f : S^n \to \mathbb{R}^n$. Define $g : S^n \to \mathbb{R}^n$ by $g(x) = f(x) - f(-x)$. Then $g$ is an antipodal map, so by (1b) there exists a point $y \in S^n$ such that $g(y) = 0$. Therefore $f(y) = f(-y)$, as desired.

(1b) $\Rightarrow$(2a) This follows from taking the standard embedding of $S^{n-1}$ in $\mathbb{R}^n$.

(2a) $\Rightarrow$(1b) Assume that there is an antipodal map $f : S^n \to \mathbb{R}^n - \{0\}$. Define $g : S^n \to S^{n-1}$ by $g(x) = f(x)/\|f(x)\|$. Then $g$ is antipodal, completing the proof.

(2a) $\Rightarrow$(2b) Let $f : B^n \to S^{n-1}$ such that $f|_{\partial B^n}$ is antipodal. Thinking of $S^n$ and $B^n$ as embedded in $\mathbb{R}^{n+1}$ with $B^n$ having $(n+1)$st coordinate zero, we define a map $g : S^n \to S^{n-1}$ by:

$$g(x_1, x_2, \ldots, x_n, x_{n+1}) = \begin{cases} f(x_1, x_2, \ldots, x_n) & \text{if } x_{n+1} \geq 0 \\ -f(-x_1, -x_2, \ldots, -x_n) & \text{if } x_{n+1} < 0 \end{cases}$$

The assumption that $f$ is antipodal on $\partial B^n$ ensures that the map $g$ is continuous and, combined with the definition of $g$, shows that $g$ is an antipodal map.

(2b) $\Rightarrow$(2a) Let $f : S^n \to S^{n-1}$ be antipodal. Restricting $f$ to the upper hemisphere of $S^n$ gives the desired map from $B^n \to S^{n-1}$.

(1a) $\Rightarrow$(C) Let $F_1, \ldots, F_{n+1}$ be a closed cover of $S^n$. Define $f : S^n \to \mathbb{R}^n$ by

$$f(x) = (d(x, F_1), d(x, F_2), \ldots, d(x, F_n))$$

Then by (1a) there exists a point $x_0 \in S^n$ such that $f(x_0) = f(-x_0)$. Let $y = (y_1, \ldots, y_n) := f(x_0)$. If any of the $y_i$ are zero then $x_0 \in F_i \cap (-F_i)$ and we are done. So assume that all of the $y_i \neq 0$. Then since the $F_i$ are a cover of $S^n$, $x_0$ and $-x_0$ are in $F_{n+1}$, completing the proof.

(C) $\Rightarrow$(O) Let $U_1, \ldots, U_{n+1}$ be an open cover of $S^n$. Then for all $x \in S^n$ there exists an neighborhood $V_x$ of $x$ such that $\mathrm{cl}(V_x) \subseteq U_i$ for some $i$. By compactness we can pick some finite subset $A \subseteq S^n$ such that the $V_x$ for $x \in A$ cover $S^n$. Let $F_i := \cup\{\mathrm{cl}(V_x) : x \in A, \mathrm{cl}(V_x) \in U_i\}$. Then the $F_i$ are a closed cover of $S^n$ and by (C) we know that there exists some $j$ such that $F_j \cap (-F_j) \neq \emptyset$. Since $F_j \subset U_j$ we have the desired result.

(O) $\Rightarrow$(C) Let $F_1, \ldots, F_{n+1}$ be a closed cover of $S^n$. Define $U_i^\epsilon := \{x \in S^n : d(x, F_i) < \epsilon\}$. By (O), for each $\epsilon > 0$ there exists an $i_\epsilon$ and $x^\epsilon$ such that $\pm x^\epsilon \in U_{i_\epsilon}^\epsilon$. Now let $\epsilon_n = \frac{1}{n}$. There exists some $j$ such that $i_{\epsilon_n} = j$ for infinitely many $n \in \mathbb{N}$. By compactness, we have a convergent subsequence of points $x_{n_m}$ such that $\pm x_{n_m} \in U_j^{\epsilon_{n_m}}$. Let $x$ be the limit of the $x_{n_m}$. Then $\pm x \in \cap_m U_j^{\epsilon_{n_m}} = F_j$.

(C) $\Rightarrow$(2a) We first show that there exists a closed cover $G_1, \ldots, G_{n+1}$ of $S^{n-1}$ such that $G_i \cap (-G_i) = \emptyset$ for all $i$. To do this, embed the $n$-simplex as a convex hull of $n+1$ points in $\mathbb{R}^n$ such that all of the points are in the interior of $B^{n-1}$ and the origin is in the interior of the $n$-simplex. Label the facets of the $n$-simplex by $H_1, \ldots, H_n$. Then let $x \in G_i$ if the ray $\overrightarrow{0x}$ intersects $H_i$.

Now assume there exists an antipodal map $f : S^n \to S^{n-1}$. Then the sets $F_i := f^{-1}(G_i)$ provide a counterexample to (C). $\square$

## 2.2 Proofs of the Borsuk-Ulam Theorem

We begin by sketching two proofs of the Borsuk-Ulam theorem, one homological and one geometrical. We then give a detailed combinatorial proof.

### 2.2.1 A Homological Proof

We prove version (2a) of the theorem by contradiction. Assume that there is an antipodal map $f : S^n \to S^{n-1}$. If we mod out by the $\mathbb{Z}_2$ action on both spheres we get a map $f : \mathbb{R}P^n \to \mathbb{R}P^{n-1}$. This induces a map on cohomology, $f_* : H^\bullet(\mathbb{R}P^{n-1}, \mathbb{Z}_2) \to H^\bullet(\mathbb{R}P^n, \mathbb{Z}_2)$. Recall that $H^\bullet(\mathbb{R}P^{n-1}, \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^n)$ and $H^\bullet(\mathbb{R}P^n, \mathbb{Z}_2) \cong \mathbb{Z}_2[y]/(y^{n+1})$. A computation shows that $f_*(x) = y$. Then by the naturality of the cup product we have the contradiction

$$0 = x^n = x \smile x \smile \cdots \smile x \xrightarrow{f_*} y \smile y \smile \cdots \smile y = y^n \neq 0.$$

### 2.2.2 A Geometric Proof

We will prove version (1b) of the theorem by contradiction. Assume that $f : S^n \to \mathbb{R}^n$ with $f^{-1}(0) = \emptyset$. Let $g : S^n \to \mathbb{R}^n$ be the map $g(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$. So $g^{-1}(0) = (0, 0, \ldots, 0, \pm 1)$. Define the function $F : S^n \times [0, 1] \to \mathbb{R}^n$ by $F(x, t) = (1 - t)f(x) + t(g(x))$. Note that $F$ has the following properties:

- $F|_{S^n \times \{1\}} = g$
- $F|_{S^n \times \{0\}} = f$
- For all $x \in S^n$ and $t \in [0, 1]$, $F(-x, t) = -F(x, t)$.

Now consider $F^{-1}(0)$. In case where $F$ is smooth and 0 is a regular value we know that $F^{-1}(0)$ is a one-manfold with the boundary of $F^{-1}(0)$ being the intersection of $F^{-1}(0)$ with the boundary of $S^n \times [0, 1]$. From our knowledge of $f$ and $g$ we have that $F^{-1}(0)$ intersects the boundary of $S^n \times [0, 1]$ only at two antipodal points of $S^n \times \{1\}$. So one component of $F^{-1}(0)$ is a closed segment with endpoints in $S^n \times \{1\}$ but no other points in $S^n \times \{1\}$. In order to have $F(-x, t) = -F(x, t)$, if $(x, t)$ is on this segment then $(-x, t)$ must also be on this segment. This is impossible if our segment is a connected one-manifold (with boundary).

In Matoušek's book one can find the details of this argument, as well as a description of the approximations required to deal with the fact that $F$ need not be a smooth map.

### 2.2.3 A Combinatorial Proof

We now show that the Borsuk-Ulam theorem, in particular formulation (2b), is equivalent to a combinatorial statement called Tucker's lemma. We will then provide a combinatorial proof of Tucker's lemma.

**Lemma 2.2** (Tucker's Lemma)**.** *Let $T$ be a triangulation of $B^n$ which is antipodally symmetric on $\partial B^n$. Let $\lambda : V(T) \to \{\pm 1, \pm 2, \ldots, \pm n\}$ such that $\lambda(-v) = -\lambda(v)$ for all vertices $v$ on $\partial B^n$. Then there exists an edge $\{u, v\} \in T$ such that $\lambda(u) = -\lambda(v)$.*

The $n$-dimensional cross polytope is conv$\{\pm e_i : e_i$ unit vectors in $\mathbb{R}^n$ $\}$. Let $\diamond^{n-1}$ be the boundary complex of the $n$-dimensional cross polytope, hence $\diamond^{n-1} \cong S^{n-1}$. Equivalently, $\diamond^{n-1}$ is the join of $n$ copies of $S^0$. The vertices of $\diamond^{n-1}$ are $V(\diamond^{n-1}) = \{\pm i : 1 \leq i \leq n\}$. The faces of $\diamond^{n-1}$ are all of the subsets of the vertices that do not contain any antipodal pairs. Therefore, for a triangulation $T$ as in Tucker's lemma, the following statement is equivalent to Tucker's Lemma:

$$\text{There does not exist a simplicial map } \lambda : T \to \diamond^{n-1} \text{ which is antipodal on } T \cap \partial B^n. \tag{2.1}$$

If there did exist a map $\lambda$ that violated condition (2.1) then the map $\|\lambda\| : \|T\| \cong B^n \to \|\diamond^{n-1}\| \cong S^{n-1}$ would violate equivalent statement (2b) of the Borsuk-Ulam theorem. Hence the Borsuk-Ulam theorem implies condition (2.1) and Tucker's Lemma. We now prove the reverse implication.

*Proof that Tucker's Lemma implies the Borsuk-Ulam theorem (2b).* Assume that there is a map $f : B^n \to S^n$ such that $f|_{\partial B^n}$ is antipodal. We will use $f$ to construct a triangulation $T$ and map $\lambda$ that satisfy the conditions of Tucker's Lemma and then use Tucker's lemma to obtain a contradiction.

Let $\epsilon = 1/\sqrt{n}$. For all $y \in S^{n-1}$ we know that $\|y\|_\infty \geq \epsilon$. Since $B^n$ is compact we know that $f$ is uniformly continuous. Therefore, there exists a $\delta > 0$ such that for all points $x, x' \in B^n$ with $d(x, x') < \delta$ we have $\|f(x) - f(x')\|_\infty < 2\epsilon$. Let $T$ be a triangulation of $B^n$ with diam$(T) < \delta$, i.e. if $\{u, v\} \in T$ then $d(u, v) < \delta$. For all $v \in V(T)$, let $k(v) = \min\{j : |f(v)_j| \geq \epsilon\}$. Define a map $\lambda(v)$ on $V(T)$ by

$$\lambda(v) := \begin{cases} +k(v) & \text{if } f(v)_{k(v)} > 0 \\ -k(v) & \text{if } f(v)_{k(v)} < 0 \end{cases}$$

The fact that $f$ is antipodal means that $\lambda$ is also antipodal.

Applying Tucker's Lemma, there exists an edge $\{u, v\} \in T$ such that $\lambda(v) = -\lambda(u) > 0$. Therefore $k(v) = k(u)$; define $i$ to be this common value. Then $f(v)_i \geq \epsilon$ and $f(u)_i \leq \epsilon$. Hence $\|f(u) - f(v)\|_\infty \geq 2\epsilon$, contradicting our uniform continuity condition.

□

We next give a proof of a special case of Tucker's Lemma. Fix $n > 0$. Let

$$H_k^+ = \{x \in S^{n-1} : x_{k+1} \geq 0, x_{k+2} = x_{k+3} = \cdots = x_n = 0\}$$
$$H_k^- = \{x \in S^{n-1} : x_{k+1} \leq 0, x_{k+2} = x_{k+3} = \cdots = x_n = 0\}$$

We will restrict our proof to the case of triangulations $T$ such that every $H_k^\pm$ is triangulated by a subcomplex of $T$, $0 \leq k \leq n-1$. We can construct such $T$ with arbitrarily small diameter, hence the proof of this special case of Tucker's Lemma will still be sufficient to prove the Borsuk-Ulam Theorem.

*Proof of a Special Case of Tucker's Lemma (Lemma 2.2).* We work with coefficients in $\mathbb{Z}_2$. For a simplicial complex $\Delta$ , let $\Delta_k$ be the set of all $k$-simplexes of $\Delta$ . Let $C_k(\Delta) := \oplus_{F \in \Delta_k}(\mathbb{Z}_2 \cdot F)$ be the set of all $k$-chains in $\Delta$ . Then we have a one to one correspondence between sets $\{F_1, \ldots, F_m\} \in 2^{\Delta_k}$ and $k$-chains $\sum_{i=1}^m 1 \cdot F_i \in C_k(\Delta)$.

We want to define the boundary map $\partial_k : C_k(\Delta) \to C_{k-1}(\Delta)$ (the subscript on $\partial_k$ will often be omitted when the dimension of the domain is clear). We start by defining $\partial$ on a single $k$-face $F \in \Delta$:

$$\partial F := \sum_{v \in F} 1 \cdot (F - \{v\})$$

For a $k$-chain $C_k = \sum_{i=1}^m F_i$ we define $\partial C_k := \sum_{i=1}^m \partial F_i$. If $C_k$ and $D_k$ are $k$-chains then

$$\partial(C_k + D_k) = \partial C_k + \partial D_k \qquad \text{and} \qquad \partial(\partial C_k)) = 0.$$

From our definition of $\partial$, to prove these results it is sufficient to check them in the case where $C_k = 1 \cdot F_i$ and $D_k = 1 \cdot F_j$. These straightforward calculations are left to the reader.

Now let $K$ and $L$ be simplicial complexes and let $f : K \to L$ be a simplicial map. Then we have an induced map $f_{\sharp k} : C_k(K) \to C_k(L)$ defined on simplexes by

$$f_{\sharp k}(F) = \begin{cases} \{f(v) : v \in F\} & \text{if } f|_F \text{ is injective} \\ 0 & \text{otherwise} \end{cases}$$

We extend $f_{\sharp k}$ linearly to all of $C_k(K)$. Note that $\partial_{k-1} f_{\sharp k} = f_{\sharp(k-1)} \partial_k$. Once again, it is sufficient to prove this on a single simplex $F$. The proof can be broken up into cases depending on the cardinality of the image of $F$ under $f$. Note that in all of the cases where $f|_F$ is not injective, both sides of the equality are zero.

**Claim:** Let $K = T \cap \partial B^n$ (so $K$ is antipodally symmetric). Let $L$ be another antipodally symmetric triangulation of $S^{n-1}$. Let $f : K \to L$ be a simplicial map. Let $A_{n-1} := \sum_{F \in K_{n-1}} 1 \cdot F$ and $C_{n-1} := f_\sharp(A_{n-1})$. Then:

1. Either $C_{n-1} = 0$ or $C_{n-1} = \sum_{\sigma \in L_{n-1}} 1 \cdot \sigma$. In these two cases we say that $\deg_2(f) = 0$ or $\deg_2(f) = 1$ respectively.

2. If $\bar{f} : T \to L$ is a simplicial map such that $\bar{f}|_K = f$ then $\deg_2(f) = 0$.

3. If $f : K \to L$ is antipodal then $\deg_2(f) = 1$.

Taking $L = \diamond^{n-1}$, parts (2) and (3) of this claim imply condition (2.1) which in turn implies Tucker's Lemma for the triangulations under consideration. So once we prove the claim we will have shown the desired result.

**Proof of (1):** Note that every face $G \in L_{n-2}$ is contained in exactly two elements of $L_{n-1}$. Assume that $C_{n-1} \neq 0$ and $C_{n-1} \neq \sum_{\sigma \in L_{n-1}} 1 \cdot \sigma$. Then there exists a face $F \in L_{n-2}$ such that $F \subseteq H_1, H_2$ where $H_1 \in C_{n-1}$ and $H_2 \in L_{n-1} \backslash C_{n-1}$. Then $\partial f_\sharp(A_{n-1}) = \partial C_{n-1}$ contains $1 \cdot F$ as a summand. However, $f_\sharp \partial(A_{n-1}) = f_\sharp(0) = 0$. This contradicts the commutativity of $f_\sharp$ and $\partial$.

**Proof of (2):** Let $A_n = \sum_{F \in T_n} 1 \cdot F$. Since $\partial A_n = A_{n-1}$ and $\bar{f}$ can not be injective on any $n$-simplex we have:

$$C_{n-1} = f_\sharp A_{n-1} = \bar{f}_\sharp A_{n-1} = \bar{f}_\sharp \partial A_n = \partial \bar{f}_\sharp A_n = \partial(0) = 0$$

**Proof of (3):** Define $A_k^+$ and $A_k^-$ by:

$$A_k^\pm = \sum_{F \in K_k \cap H_k^\pm} 1 \cdot F$$

Define $A_k := A_k^+ + A_k^-$. Note that for $k = n-1$ this agrees with our previous definition of $A_{n-1}$. We know that $\partial(A_k^+) = A_{k-1} = \partial(A_k^-)$. This gives $\partial(A_k) = 0$.

Define $C_k^+$ and $C_k^-$ by $C_k^\pm := f_\sharp(A_k^\pm)$. Similarly, define $C_k := f_\sharp(A_k) = C_k^+ + C_k^-$. Then

$$\partial C_k^\pm = \partial f_\sharp A_k^\pm = f_\sharp \partial A_k^\pm = f_\sharp A_{k-1} = C_{k-1} \tag{2.2}$$

Assume that $C_{n-1} = 0$; we will show that this assumption results in a contradiction, proving the desired claim. We know

$$C_{n-1} = f_\sharp A_{n-1} = f_\sharp(A_{n-1}^+ + A_{n-1}^-) = f_\sharp(A_{n-1}^+) + f_\sharp(A_{n-1}^-) = C_{n-1}^+ + C_{n-1}^-$$

Combining this with our assumption yields $C_{n-1}^+ = C_{n-1}^-$. Since $f$ is antipodal we know that $C_{n-1}^+$ and $C_{n-1}^-$ are antipodal. Therefore $C_{n-1}^+$ is antipodally symmetric. Define $D_{n-1} := C_{n-1}^+ = C_{n-1}^-$. So $D_{n-1}$ is an antipodally symmetric chain with $\partial D_{n-1} = \partial(C_{n-1}^+) = C_{n-2}$.

We now show that every $C_k$, $0 \leq k \leq n-2$ is the boundary of an antipodally symmetric chain. Our proof is by reverse induction on $k$. The above work establishes the base case $k = n-2$. We now prove the inductive step.

Assume that there exists an antipodally symmetric $(k+1)$-chain $D_{k+1}$ such that $\partial D_{k+1} = C_k$. Decompose $D_{k+1}$ into two parts $D_{k+1} = E^a_{k+1} + E^b_{k+1}$ such that $E^a_{k+1}$ is antipodal to $E^b_{k+1}$. Then we have

$$C_k^+ + C_k^- = C_k = \partial D_{k+1} = \partial(E^a_{k+1} + E^b_{k+1}) = \partial(E^a_{k+1}) + \partial(E^b_{k+1})$$

$$C_k^+ + \partial(E^a_{k+1}) = C_k^- + \partial(E^b_{k+1}). \tag{2.3}$$

Note that the two sides of equation (2.3) are antipodal. Therefore $D_k := C_k^+ + \partial(E^b_{k+1})$ is an antipodally symmetric set. Furthur, combining the fact that $\partial^2 = 0$ with (2.2) we have

$$\partial D_k = \partial(C_k^+ + \partial(E^b_{k+1})) = \partial C_k^+ = C_{k-1}.$$

This completes the inductive step.

Now consider the case $k = 0$. By the above work there exists an antipodally symmetric 1-chain $D_1$ such that $\partial D_1 = C_0$. The boundary of any antipodally symmetric 1-chain must be an even number of pairs of antipodal points. However, $C_0$ is a single pair of antipodal points, giving us the desired contradiction. □

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