The nature and significance of Gödel's incompleteness theorems

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What Gödel accomplished in the decade of the 1930s before joining the Institute changed the face of mathematical logic and continues to influence its development. As you gather from my title, I'll be talking about the most famous of his results in that period, but first I want to indulge in some personal reminiscences. In many ways this is a sentimental journey for me. I was a member of the Institute in 1959-60, a couple of years after receiving my PhD at the University of California in Berkeley, where I had worked with Alfred Tarski, another great logician. The subject of my dissertation was directly concerned with the method of arithmetization that Gödel had used to prove his theorems, and my main concern after that was to study systematic ways of overcoming incompleteness. Mathematical logic was going through a period of prodigious development in the 1950s and 1960s, and Berkeley and Princeton were two meccas for researchers in that field. For me, the prospect of meeting with Gödel and drawing on him for guidance and inspiration was particularly exciting. I didn't know at the time what it took to get invited. Hassler Whitney commented for an obituary notice in 1978 that "it was hard to appoint a new member in logic at the Institute because Gödel could not prove to himself that a number of candidates shouldn't be members, with the evidence at hand." That makes it sound like the problem for Gödel was deciding who *not* to invite. Anyhow, I ended up being one of the lucky few.

Once I got settled, my meetings with Gödel were strictly regulated affairs. There were few colleagues with whom he had extensive contact--Albert Einstein, as everyone knows, and Oskar Morgenstern among them, and a few of the more senior visiting logicians. There were some younger logicians who managed to see him more often than I did, but I was too intimidated to take full advantage of him, something I regret to this day. Gödel's office was directly above the one that I shared with another visitor, the Japanese logician, Gaisi Takeuti. We used to think we heard him pacing the floor above us. When I wanted to meet Gödel and figured he was in his office, I'd phone him for an appointment and would hear the phone ring and hear him answer. When it worked out, I would walk upstairs to his office. There he would be seated at his desk, and I would sit down across from him; there was no work at the blackboard as is common among mathematicians. It was clear to me that Gödel had read my papers and knew about the work in progress, and the latter would be the focus of our conversations. He would raise some questions and make a few suggestions, and what he had to say would be very much to the point and fruit for further thought. After precisely half an hour the alarm on his watch would go off, and he would say, "I have to take my pills." I took that as my cue to leave.

The director of the Institute at that time was J. Robert Oppenheimer, and he arranged a short welcoming meeting with each new member. In my meeting with him, I had in mind that I just might mention our contact--of a sort--a dozen years earlier when, as a student waiter in the faculty center at CalTech, I had served him lunch one day, but the right moment never presented itself. In our conversation, I was surprised how much Oppenheimer knew and understood about the work that had brought me to the Institute. I was reminded of that some twenty years later when I was a visiting fellow at All Souls College in Oxford. One evening a dinner was held at which the guest of honor was Margaret Thatcher. In the informal gathering afterward she circulated among all the fellows and visitors; when my turn came, within two minutes she had sized up what I was doing to her satisfaction and then told me what I *ought* to be doing. Well, Oppenheimer didn't go *that* far, but I bet he could have.

To get back to Gödel, of the three major results that he obtained in mathematical logic in the 1930s, only the incompleteness theorem has registered on the general consciousness, and inevitably popularization has led to misunderstanding and misrepresentation. Actually, there are two incompleteness theorems, and what people have in mind when they speak of Gödel's theorem is mainly the first of these. Like Heisenberg's Uncertainty Principle, it has captured the public imagination with the idea that there are absolute limits to what can be known. More specifically, it's said that Gödel's theorem tells us there are mathematical truths that can never be proved. Among postmodernists it's used to support skepticism about objective truth; nothing can be known for sure. And in the *Bibliography of Christianity and Mathematics* (yes, there is such a publication!) it's asserted that "theologians can be comforted in their failure to systematize revealed truth because mathematicians cannot grasp all mathematical truths in their systems either." Not only that, the incompleteness theorem is held to imply the existence of God, since only He can decide all truths.

Among those who know what the incompleteness theorems actually do tell us, there are some interesting views about their wider significance for both mind and matter. In his 1960 Gibbs Lecture, Gödel himself drew the conclusion that "either mind infinitely surpasses any finite machine or there are absolutely unsolvable number theoretic problems." He evidently believed that mind can't be explained mechanically, but since he couldn't give an unassailable argument for that--in his typical style he formulated this in a more cautious way as a dichotomy. A lot has been written pro and con about the possible significance of Gödel's theorem for mechanical models of the mind by a number of logicians and philosophers; my own critique of Gödel's dichotomy is published in the July 2006 issue of the journal *Philosophia Mathematica,* that you can also find on my home page.[1] One of the most prominent proponents of the claim that Gödel's theorem proves that mind is *not* mechanical is Roger Penrose (e.g. in *Shadows of the Mind*): "there must be more to human thinking than can ever be achieved by a computer". However, he thinks that there must be a scientific explanation of how the mind works, albeit in its non-mechanical way, and that ultimately must be given in physical terms, but that current physics is inadequate to do the job. As far as I know, Penrose does not say that Gödel's theorem puts any limits on what one may hope to arrive at in the search for those needed new laws of physics. But Stephen Hawking and Freeman Dyson, among others, have come to the conclusion that Gödel's theorem implies that there can't be a Theory of Everything. Both the supposed consequences of the incompleteness theorem

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[1] http://math.stanford.edu/~feferman/papers.html . You can also find earlier discussions referenced there.

for the nature of mind and the laws of the universe are quite interesting and should be examined on their own merits. A discussion of the first of these here would take us too far afield. But I do want to return later to say something about the argument contra any putative TOE.

It's time to say something a little more precise about just what the incompleteness theorems are. For the non-experts who want to delve further into them, I strongly recommend two recent books, both by Torkel Franzén, who unfortunately died in mid-life last spring of cancer:

Inexhaustibility: A non-exhaustive treatment, is for readers with a moderate amount of logical and mathematical background.
Gödel's Theorem. An incomplete guide to its use and abuse, is for the general reader. Both are published by A. K. Peters.

Let's start with a current formulation of Gödel's first incompleteness theorem that is imprecise but can be made precise:

In any sufficiently strong formal system there are true arithmetical statements that can't be proved (in the system).

Practically everything in this formulation needs to be elaborated. First of all, we have to explain what is meant by a *formal system*, and to do that we need to say what is meant by a *formal language* L. For that we have to prescribe a list of basic symbols and we have to say which finite sequences of basic symbols constitute meaningful expressions of the language, and we have to do this in a way that can be checked by a computer, at least in principle.

This is illustrated by *the formal language for arithmetic*. When mathematicians speak of arithmetic they mean what can be established in general about the whole numbers, more technically the positive integers 1, 2, 3,…,

--not arithmetic in the sense of grade school rules of calculation. The study of this higher arithmetic is also called number theory. In the formal language for arithmetic we have a symbol for 1, symbols for addition, +, and multiplication, ×, symbols 'n', 'm',… that act as variables interpreted as referring to arbitrary (positive) integers, symbols for the equality and less-than relations, = and <, and symbols for the logical particles 'and' (&), 'or' (∨), 'not' (¬), 'implies' (⇒), 'if and only if' (⇔) and what are called the quantifiers, 'for all n' (∀n) and 'there exists n' (∃n) , as well as parentheses to avoid ambiguous expressions. In this language we can express that a number m is *prime*—Prime(m)--by saying that m is greater than 1 and there do not exist n and k between 1 and m whose product is n:

$$1 < m \;\&\; \neg(\exists n)(\exists k)(1 < n < m \;\&\; n \times k = m\,).$$

There are many interesting problems in number theory that concern the prime numbers. The first few primes are

2, 3, 5, 7, 11, 13, 17, 19, 23,…

The statement that there are infinitely many prime numbers can be expressed in our formal language in the form

$$(\forall n)(\exists m)\,(n < m \;\&\; \mathrm{Prime}(m)).$$

This is in fact true, and was known to the Greeks; a proof can be found in Euclid's *Elements*. Also proved there is that the prime numbers are another kind of building block for the positive integers, since every number greater than 1 can be written as a product of one or more prime numbers, in one and only one way in order of size.

Mathematicians working on arithmetical problems became interested in primes with more special properties, for example what are called *twin prime pairs*, namely integers m and m + 2 both of which are prime, like 3 and 5 or 11 and 13. Note that m = 7 and m = 13 don't work as the first terms of twin prime pairs. It is conjectured that there are infinitely many twin primes, i.e. that

$$(\forall n)(\exists m)(n < m \;\&\; \mathrm{Prime}(m) \;\&\; \mathrm{Prime}(m + 2)).$$

A lot of work has gone into settling that conjecture, but to this day nobody knows whether it is true or not. Another old speculation that can be expressed in the formal language for arithmetic is Goldbach's conjecture, that every even number greater than 4 is the sum of two odd primes, and we don't know whether that is true or false, either. There are also problems that are originally stated in the language of the real and complex number systems used in the calculus, that turn out to be equivalent to problems that can be stated in our language of arithmetic. One such is the Riemann Hypothesis (RH), a statement formulated by the brilliant mathematician Bernhard Riemann in the 19th century. RH is one of the seven millennium prize problems set by the Clay Mathematics Institute, the solution of any one of which would be rewarded by a million dollar prize. So this shows that already in the formal language of arithmetic one can state very important problems that remain unsolved despite considerable efforts to determine which way they go.

Once we have set up a formal language L, we can specify a formal system S in L by telling which sentences A of L are *axioms* and which relations between sentences are *rules of inference*. As it happens, from the work of Gödel on the completeness of a system of pure logic, there is a standard set of axioms and rules of inference that suffice for all logical deductions and that can be specified once and for all. So to fix a formal system S we have only to tell which are its non-logical axioms, i.e. the axioms that are specific to its subject matter. Moreover, it is also required that we should be able to check by a computer (again in principle) which sentences are axioms and which sentence combinations count as applications of the rules of inference. By a *proof* in S is meant a finite sequence of sentences, each of which is either an axiom or is obtained from earlier sentences in the list by application of one of the rules of inference. The sentence A is said to be *provable in S,* if there is a proof in S which ends with A. S is said to be *consistent* if there is no sentence A such that both A and not-A are provable in S. There are two concepts of completeness related to Gödel's theorem: (i) S is said to be *formally complete* for L if for every sentence A of L, either A is provable in S or not-A is provable in S. (In Gödel's terminology, every sentence of L is *decided* by S. If S is not complete then there are *undecidable* L-*sentences* in S.) (ii) S is *truth complete for* L if every true sentence of L is provable in S. (This terminology is not standard.) If S is consistent and truth complete for L it is formally complete, because each sentence A of L is either true or false, i.e. its negation is true. But S may be consistent and formally complete and not truth complete, because it may prove false sentences; this is a consequence of Gödel's theorem. It is of course stronger to prove of a given S that it is not formally complete, than that it is not truth complete, and that's what Gödel did in his original formulation. But there is a slight technical complication about his argument that I wanted to avoid here, which is why we'll concentrate on the weaker incompleteness result.

Taking the concept of truth in the integers for granted, we now understand everything in the above rough formulation of Gödel's first incompleteness theorem except what it means to be "sufficiently strong". Usually that is taken to mean that S includes the system PA of *Peano Arithmetic*;[2] that is a formal version of the axioms proposed for arithmetic by the Italian mathematician Giuseppe Peano in the 1890s. Its axioms assert some simple basic facts about addition, multiplication and the equality and less-than relations. Beyond those, its main axioms are all the instances of the principle of mathematical induction that can be expressed in the language of arithmetic, namely:

$$P(1)\ \&\ (\forall n)(P(n) \Rightarrow P(n+1)) \Rightarrow (\forall n)P(n)\ .$$

We can now formulate one current precise version of Gödel's first incompleteness theorem as follows:

<u>The first incompleteness theorem</u>. If S is a formal system such that

[2] Actually, much weaker systems than PA are known to suffice for the theorem.

(i) the language of S contains the language of arithmetic,

(ii) S includes PA, and

(iii) S is consistent

then there is an arithmetical sentence A which is true but not provable in S.

Here is an idea of how Gödel proved his incompleteness theorem. He first showed that a large class of relations that he called recursive, and that we now call *primitive recursive*, can all be defined in the language of arithmetic. Moreover, *every numerical instance of a primitive recursive relation is decidable in PA*. Similarly for primitive recursive functions. Among the functions that are primitive recursive are exponentiation, factorial, and the prime power representation of any positive integer.

He then attached numbers to each symbol in the formal language L of S and, using the product-of-primes representation, attached numbers as codes to each expression E of L, considered as a finite sequence of basic symbols. These are now called the *Gödel number* of the expression E. In particular, each sentence A of L has a Gödel number. Proofs in S are finite sequences of sentences, and so they too can be given Gödel numbers. He then showed that the property:

n is the number of a proof of A in S,

written

$\mathrm{Proof}_S(n, A)$

is primitive recursive and so expressible in the language of arithmetic.[3] Hence the sentence

$(\exists n)\ \mathrm{Proof}_S(n, A)$,

written

$\mathrm{Prov}_S(A)$

[3] To be more precise, $\mathrm{Proof}_S(n, A)$ is here written for $\mathrm{Proof}_S(n, m)$, where m is the Gödel number of A.

expresses that A is provable from S. Moreover, if it is true, it is provable in PA. So we can also express directly from this that A is *not* provable from S, by $\neg \mathrm{Prov}_S(A)$. Finally, Gödel used an adaptation of what is called *the diagonal method* to construct a specific sentence, call it D, such that PA proves:

$$D \Leftrightarrow \neg \mathrm{Prov}_S(D).$$

Finally, he showed:

(*) If S is consistent then D is not provable from S.

The argument for (*) is by contradiction: suppose D *is* provable from S. Then we could actually produce an n which is a number of a proof in S of D, and from that we could prove in PA that "n is the number of a proof of D in S", from which follows "D is provable in S". But this last is equivalent in S to $\neg D$, so S would be inconsistent, contradicting our hypothesis. Finally, the sentence D is true because it is equivalent, in the system of true axioms PA, to the statement that it is unprovable from S.

It should be clear from the preceding that the statement that S is consistent can also be expressed in the language of arithmetic, as $\neg \mathrm{Prov}_S(A \& \neg A)$, for some specific A (it does not matter which); we write Con_S for this. Then we have:

The second incompleteness theorem. If S is a formal system such that
(i) the language of S contains the language of arithmetic,
(ii) S includes PA, and
(iii) S is consistent,
then the consistency of S, Con_S is not provable in S.

The way Gödel established this is by formalizing the entire preceding argument for the first incompleteness theorem in Peano Arithmetic. It follows that PA proves the formal expression of (*), i.e. it proves

(**) $\mathrm{Con}_S \Rightarrow \neg \mathrm{Prov}_S(D)$.

But by the construction of D, it follows that PA (and hence S) proves

(***) $\mathrm{Con}_S \Rightarrow D$.

Thus if S proved Con_S it would prove D, which we already know to be not the case.

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There are several directions in which to explore the significance of the incompleteness theorems. But first, a bit of history. As John Dawson reminded me, by coincidence November 17, 1930 is the date of receipt for publication of "Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme I" [On formally undecidable propositions in *Principia Mathematica* and related systems, Part I] by the journal *Monatshefte für Mathematik und Physik*. This is the paper in which the first incompleteness theorem was proved in full for a certain class of formal systems and in which the second incompleteness theorem was announced, for publication. (It appeared there soon after submission, in January 1931.) Gödel promised a Part II in which a proof of the second theorem would be given but that never appeared. He said it was because his results were accepted so quickly but, as we shall see, that was by no means generally the case.

Gödel had made an informal announcement of the first incompleteness theorem at a meeting on the philosophy of science and the foundations of mathematics at Königsberg, Germany in September 1930. Except for John von Neumann, this was met with general incomprehension by the scholars in attendance. Von Neumann was very impressed with the result. Thinking about it a couple of months after the meeting, he realized that the second incompleteness theorem could be derived from the argument for the first and communicated that in a letter to Gödel dated Nov. 20, 1930—as it happens, three days after the acceptance of Gödel's article by the *Monatshefte*. Gödel sent von Neumann a copy of the article, and of course von Neumann readily accepted Gödel's priority. More importantly, both Gödel and von Neumann realized that the second incompleteness theorem was of direct significance for Hilbert's program in the foundations of mathematics, though just in what way was initially a matter of dispute between them, and then between them and Hilbert. Here's the background to that.

David Hilbert, as we all know, was one of the most important mathematicians of the time, having made fundamental contributions to practically all the main areas of pure mathematics as well as applications of mathematics to physics. Off and on from the end of the 19^{th} century through the first third of the 20^{th} century he was also very much concerned with the foundations of mathematics and was troubled by contradictions found by Cantor and Russell in the theory of sets, the most general mathematical theory to have been developed for foundational purposes.

Hilbert's idea was to secure the foundations of mathematics on a solid basis, and to do this in a convincing way. He proposed to model mathematical reasoning in formal systems so as to be able to establish precise results about them. Formal systems are an idealized model of mathematical reasoning; in practice, mathematicians don't use strictly limited formal languages in which to carry out their reasoning and don't appeal explicitly to axioms or rules of inference to justify their arguments. But the work on formalization showed that mathematical reasoning as it is actually carried out can be faithfully represented in suitable formal systems. Of course, one needs richer languages to include variables ranging over different kinds of number systems like the real numbers and the complex numbers, and over sets and functions of various types, in order to do that. A minimal criterion for the acceptability of a formal system S is that it should be consistent. After all, if S is to be used to obtain truths about numbers and other kinds of mathematical objects, it should start from true axioms and use rules of inference that invariably lead from truths to truths. In that case, we would never be able to prove a statement of the form $(A \,\&\, \neg A)$ from S. So Hilbert's first aim was to show for stronger and stronger systems for mathematics, beginning with arithmetic, that they are consistent. Because the concept of a formal system is explained in precise mathematical terms, the question of its consistency is a precise mathematical problem. Looked at in this way, *Hilbert wanted to apply mathematics to secure mathematics*, but he conceived of the enterprise as a new subject that he called *metamathematics*.

However, it would be circular for this purpose to allow unrestricted mathematical means in metamathematics. In particular, according to Hilbert, the metamathematical program should completely eschew concepts and principles that involve the *actual* or *completed infinite*. He believed that the contradictions like Cantor's and Russell's paradoxes had their source in the essential use of such concepts. (In fact he was wrong about that, but that's another story.) Hilbert concluded that in order to prove the consistency of various formal systems for mathematics that embody the actual infinite, one must use *only* concepts and reasoning dealing solely with finite objects in a completely finitary way. However, Hilbert did not make precise exactly what methods are *finitary* (also called *finitistic*), but only gave examples. This later became the point of contention, as I'll explain.

Peano Arithmetic provided the first real challenge for Hilbert's program since it implicitly involves the actual infinite in its acceptance of such statements as:

$$(\forall n)A(n) \vee (\exists n)(\neg A(n)),$$

which are intuitively verified by running through the positive integers 1, 2, 3,… one after another looking to see if we ever reach an n for which A fails; we must go through all the integers if there is no such n. Gödel actually set out to prove the consistency of PA by finitistic means, but in the process met a basic obstacle, and when he analyzed the difficulty, he was led to the first incompleteness theorem. When he went on to the second incompleteness theorem he thus turned the whole thing upside down. By the way, in doing so, Gödel was the first person to fully exploit the metamathematical point of view, by working *within* the system to obtain results *about* the system.

As I've said, the second incompleteness theorem raised a fundamental problem for Hilbert's program. According to it, no sufficiently strong formal system that happens to be consistent proves its own consistency. Thus, *no formal system in which all finitistic methods can be formalized can be proved consistent by finitistic methods*. So the crucial question was whether all finitistic methods can be formalized in Peano Arithmetic. After stating the second incompleteness theorem in his paper, Gödel wrote:

> I wish to note expressly that [this theorem does] not contradict Hilbert's formalistic viewpoint. For this viewpoint presupposes only the existence of a consistency proof in which nothing but finitary means of proof is used, and it is conceivable that there exist finitary proofs that *cannot* be expressed in the formalism … (*Gödel 1931, Collected Works* Vol. I , p. 195)

But von Neumann, who had already contributed to Hilbert's consistency program for a weaker system than PA, was convinced that *Peano Arithmetic already encompasses all that can be done finitistically*; this became a matter of exchange between him and Gödel. At first Gödel disagreed, but within a few years *he too* came around to that conclusion. Hilbert, however, *never* accepted it. In the preface to vol. I, with Paul Bernays, of *Grundlagen der Mathematik* (1934), that was planned to be an exposition of his program and the contributions that had been made to it, he wrote:

> I would like to emphasize the following: the view, which temporarily arose and which maintained that certain recent results of Gödel show that my proof theory can't be carried out, has been shown to be erroneous. In fact that result shows only that one must utilize the finitary standpoint in a sharper way for the farther reaching consistency proofs…

Gödel and Hilbert never met or corresponded and Hilbert never acknowledged what Gödel had accomplished, despite the fact that all his co-workers in the foundations of mathematics recognized its importance. Nowadays, it is almost universally agreed that Hilbert's program as originally conceived is already blocked at arithmetic. In its place, much work has gone into extended (or relativized) forms of Hilbert's program using constructive though non-finitistic means for consistency proofs.

But what about the significance of the incompleteness theorems for mathematics itself?[4] It is very tantalizing to think that the reason one hasn't been able to settle some outstanding arithmetical problems is because they are independent of the systems that embody usual methods of proof. But the proof of the incompleteness theorem sketched above doesn't tell us anything about the status of still unsolved mathematical problems, like Goldbach's conjecture or the twin prime conjecture, or the Riemann Hypothesis. In fact, so far, no unsolved problem of prior mathematical interest like these has even been shown to be independent of Peano Arithmetic. The true statement D shown to be unprovable by Gödel is just contrived to do the job; it doesn't have mathematical interest on its own. Logicians have found more natural looking arithmetical statements that are true but can't be proved from PA and even from much stronger systems S, such as variants of Ramsey's theorem and other combinatorial theorems, but none of prior interest, so the significance of that work is not clear for working mathematicians. The case of the Fermat conjecture that resisted attack for over three hundred years suggests another view of the matter: some proofs require an enormous assemblage of sophisticated and complicated mathematics and delicate argumentation to push through to the end. The Fermat conjecture (for all x, y, z positive integers and all n > 2, it is not the case that $x^n + y^n = z^n$) is stated in the language of arithmetic, but the proof by Andrew Wiles makes profound use of notions and methods that go far beyond arithmetic. Before Wiles obtained his result it was speculated that the Fermat conjecture would be an example of a statement whose truth is difficult to establish (assuming it is true) because it is independent of PA. Now that we know that it is in fact true, the question of its independence can be revisited: despite the complexity of Wiles' proof and its extensive use of non-arithmetical methods, it's not excluded that a proof can be found that is purely arithmetical. In that case, a search for an independence proof before Wiles' discovery would have come to nothing.

[4] I have written about this at greater length in my article, "The impact of Gödel's incompleteness theorems on mathematics" for the *Notices of the American Mathematical Society* 53 no. 4 (April 2006), that can also be found on my home page.

Let me return, finally, to the possible significance of the incompleteness theorems for physics, in particular for the search for a Theory of Everything. For example, in a review for the *New York Review of Books* two years ago of Brian Greene's, *The Fabric of the Cosmos*, Freeman Dyson wrote:

> Gödel's theorem implies that pure mathematics is inexhaustible. No matter how many problems we solve, there will always be other problems that cannot be solved within the existing rules. … because of Gödel's theorem, physics is inexhaustible too. The laws of physics are a finite set of rules, and include the rules for doing mathematics, so that Gödel's theorem applies to them. [*NYRB*, May 13, 2004].

So, according to Dyson's argument, there can't be a Theory of Everything.[5]

In response to that, I wrote the *NYRB* (July 15, 2004) making the following points:

1. It's indeed the case that if the laws of physics are formulated in a formal system S which includes the concepts and axioms of arithmetic as well as physical notions such as time, space, mass, charge, velocity, etc., and if S is consistent then there are propositions of higher arithmetic which are undecidable by S. But this tells us nothing about the *specifically physical laws* encapsulated in S, which could conceivably be complete *as such*.[6]

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[5] The physicist and theologian Stanley L. Jaki has been arguing the same since the 1960s; see http://pirate.shu.edu/~jakistan/JakiGodel.pdf . Jaki chides Stephen Hawking for taking so long to see the light; after having supported the pursuit of a TOE, Hawking concluded in his Dirac lecture of 2002 that there can't be such a thing on account of Gödel's theorem; see http://www.damtp.cam.ac.uk/strings02/dirac/hawking/ .

[6] Torkel Franzén makes the same point in his book *Gödel's Theorem*, p. 87. More picturesquely, he writes, "nothing in the incompleteness theorem excludes the possibility of our producing a complete theory of stars, ghosts and cats, all rolled into one, as long as what we say about stars, ghosts and cats can't be interpreted as statements about the natural numbers." He also explains (pp. 88-89) why Hawking's attempt to reduce certain problems about prime numbers to physical properties of blocks of wood is irrelevant.

2. Anyhow, all this is highly theoretical and speculative. In practice, a much different picture emerges. Beyond basic arithmetic calculations, the mathematics that is applied in physics rarely calls on higher arithmetic but depends instead mainly on substantial parts of mathematical analysis and higher algebra and geometry. All of the mathematics that underlies these applications can be formalized in the currently widely accepted system for the foundation of mathematics known as Zermelo-Fraenkel set theory, and there is not a shred of evidence that anything stronger than that system would ever be needed. Actually, it has long been recognized that much weaker systems suffice for that purpose. In fact, I have conjectured that all scientifically applicable mathematics can be formalized in a certain system that is a conservative extension of Peano Arithmetic, and there is considerable evidence in support of that conjecture.[7] Note well that the issue here only concerns *applied* mathematics. It is another matter entirely whether, and in what sense, *pure* mathematics needs new axioms beyond those of the Zermelo-Fraenkel system; that has been a matter of some controversy among logicians.

My own conclusion is that Gödel's theorem is neither here nor there concerning the fundamental laws of physics. That new theories are needed seems to be unquestioned, but whether there can be an end to the search for such is not something we can simply settle on metamathematical grounds.

To return to mathematics, whatever its relevance to practice, Gödel's theorem convincingly demonstrates the *in principle inexhaustibility of pure mathematics* in the sense of the never ending need for new axioms, and it invites us to ponder the question: *just what axioms for mathematics ought to be accepted and why*? That is really a philosophical question, and like most important philosophical questions, has no answer commanding universal agreement. Meanwhile, mathematics, like life, goes on without it.

[7] See the article, "Why a little bit goes a long way: logical foundations of scientifically appplicable mathematics", reprinted in my collection of essays, *In the Light of Logic*; the article may also be found on my home page.

However, GÃ¶delâ€™s incompleteness theorems showed that such a single theory of everything would not be possible. Not everything can be proved, as there will always be statements in mathematics that can neither be proved or disproved. GÃ¶delâ€™s First Incompleteness Theorem.Â Effect of GÃ¶delâ€™s theorem on Hilbert program. Before GÃ¶delâ€™s theorems, the mathematical world was governed by Hilbertâ€™s program. This was formulated by David Hilbert in the early 20th century to put an end to the paradoxes that were found in Set Theory. It called for a formalization of all of mathematics in axiomatic form, together with a proof that this axiomatization of mathematics is consistent. GÃ¶del's First Incompleteness Theorem states. Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetical statement that is true, but not provable in the theory (Kleene 1967, p. 250).Â Michael Dummett in the aptly titled 'The Philosophical Significance of Godel's Theorem' has argued that (G1T) may be construed as an argument against the thesis that meaning is use, by demonstrating to us that the use of any symbolic manipulation is always outrun by arithmetical truth and meaning. He introduces the notion of indefinite extensibility to salvage the thesis and provokes a lot of debate along thhe way.